Université de Bourgogne, U.F.R Sciences et Techniques
Institut de Mathématiques de Bourgogne
Ecole doctorale Carnot-Pasteur

HABILITATION À DIRIGER DES RECHERCHES

Spécialité: Mathématiques

présentée par

Emmanuel WAGNER

et soutenue publiquement le 24 novembre 2016

From categorification to topology: there and back again.

Jury composé de

Stéphane Baseilhac	Université de Montpellier	Examinateur
Anna Beliakova	Universität Zürich	Examinatrice
Christian Blanchet	Université Paris-Diderot	Rapporteur
Louis Funar	Université de Grenoble	Rapporteur
Luis Paris	Université de Bougogne	Examinateur

Au vue des rapports de

Christian Blanchet	Université Paris-Diderot
Louis Funar	Université de Grenoble
Kazuo Habiro	Kyoto University

Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand.

Albert Einstein (1879-1955)

Remerciements

Ce mémoire résume mon activité de recherche depuis mon arrivée à l'IMB. Cette activité n'aurait été ni possible ni si passionnante sans l'aide, l'amitié et la présence de plusieurs personnes.

Tout d'abord, ce manuscrit n'aurait sans doute jamais vu le jour sans les nombreux coauteurs avec qui j'ai eu le bonheur de travailler. En premier lieu ceux responsables avec moi des mathématiques de ce manuscrit que je tiens tout particulièrement à remercier: Agnès et Anne-Laure collègues et amies avant tout qui malgré les galères continuent d'avancer et de faire des mathématiques et dont le courage est une source d'inspiration perpétuelle pour moi; Benji, JB et Paolo, les autres membres des Vaskho Boyz avec qui j'espère continuer à faire des mathématiques (car on le fait si bien avec ou sans ANR!); Ivan pour sa patience et sa force de conviction; Pedro pour son inépuisable énergie mathématique; Michael pour m'avoir emmené dans son monde et j'espère sincèrement que nos projets se finaliseront bientôt!

Ensuite, je n'oublie pas non plus mes autres coauteurs: Paul, Jean-Marie et Loïc. Une mention spéciale pour Anna et Luis qui hormis le fait d'avoir collaboré avec moi m'ont permis de me lancer dans les mathématiques, ce pour quoi je les en remercie grandement ici.

J'ai une pensée aussi pour Loulou mon prochain coauteur qui outre la patience dont il a fait preuve lorsque je rédigeais ce manuscrit est responsable pour une grande partie de sa finalisation si rapide: nos projets ensemble auraient retardé d'au moins autant la finalisation de cette HDR...

Merci à Ben, Olivier et Peggy de m'amener vers le côté probabiliste des nœuds et merci à Daniele en espérant que la suite sera exceptionnelle!

Si on est toujours le ? de quelqu'un, je suis sans aucun doute le topologue d'Adrien et l'algébriste de Christian. Dans tous les cas merci à tous les deux pour les discussions dans les bureaux du troisième ou le bocal du second sur les formes normales de Smith, les recollements de tores ou les bouteilles de Klein.

L'ambiance de l'IMB n'aurait pas été aussi conviviale et stimulante pour moi sans les irréductibles de la salle café que je tiens à remercier: Ahmed, Camélia, Catherine, François, Hervé, Johan, Lucy, Nikolai, Olivier, Paolo, Pavao, Peggy, Seb et Yo.

Les mathématiques sont faites pour moi de rencontres et ce sont elles qui enrichissent nos perceptions. Il y a d'abord eu celles dans les rencontres du GDR Tresses: Catherine, Francesco, Gwenaël, Jérome, Julien, Vincent, puis celles aux WinterBraids: Delphine, Hoel, Loulou et Lukas. Merci à tous pour les bons moments passés ensemble. J'en profite d'ailleurs pour remercier Jb, Paolo et Vincent de m'avoir emmené dans l'aventure WinterBraids. On n'a jamais été aussi content d'entendre "Winter is coming".

Je tiens aussi à remercier Christian Blanchet, Louis Funar, et Kazuo Habiro qui ont accepté la tâche d'écrire un rapport sur ce mémoire. Je remercie Anna Beliakova, Stéphane Baseilhac et Luis Paris d'avoir accepté de participer au jury de cette HDR.

Je veux aussi remercier tout le personnel (passé et présent) de l'IMB et du département de mathématiques pour le support constant et précieux qu'ils assurent toujours: Anissa, Aziz, Caroline, Claire, Francis, Ibtissam, Magali, Muriel, Mylène, Nadia, Sébastien et Véronique.

Je remercie mes beaux parents pour leur disponiblité et le temps qu'ils ont su me libérer à des moments importants. Je remercie enfin ma mère et ma grand-mère, auprès de qui j'ai toujours trouvé le soutien indispensable pour avancer.

Enfin, et avant tout, toutes mes pensées sont pour Marie, Arthur et Lise qui sont depuis toujours habilités à me rendre heureux et qui le font tous les jours pour mon plus grand bonheur.

Avant-propos

Une Habilitation à diriger des recherches permet de faire le point sur les recherches effectuées depuis l'obtention du doctorat et d'en faire une présentation synthétique et c'est bien le cas dans ce manuscrit. En outre elle permet bien généralement d'exposer le point de vue de l'auteur sur un domaine de recherche ce qui est moins le cas dans ce manuscrit où l'on a préferé présenter plusieurs aspects d'un domaine de recherche révélant différentes interactions entre l'algèbre et la topologie. De manière générale, je travaille dans le domaine maintenant communément appelé *Topologie Quantique*, mais pas dans une direction très précise. Les objets que j'étudie sont de nature topologique, les nœuds, les tresses ou les surfaces mais les techniques sont de nature algébrique, topologie algébrique, combinatoire, théorie des représentations, méthodes catégoriques. Les travaux présentés dans ce manuscrit sont à l'intersection de l'algèbre et de la topologie et vous trouverez ci-dessous une liste de mes publications et remarquerez que ce manuscrit ne présente qu'une partie d'entre eux. Cette HDR est divisée en trois chapitres, chacun d'entre eux portant sur un thème précis et le choix des publications présentées est en accord avec ceux-ci. Chaque chapitre s'intéresse à une direction de recherche relativement précise dont je pense qu'elle mérite une attention particulière.

Le premier chapitre se préoccupe de l'étude des surfaces dans l'espace quatre dimensionnel, en lien avec les invariants d'entrelacs classiques et quantiques. Il fait référence d'une part à mon travail en cours avec Michael Eisermann et d'autre part à ma collaboration avec Benjamin Audoux, Paolo Bellingeri et Jean-Baptiste Meilhan:

- Slice links and the Jones polynomial, avec Michael Eisermann, in preparation.
- Homotopy classification of ribbon tubes and welded string links, avec Benjamin Audoux, Paolo Bellingeri and Jean-Baptiste Meilhan, math/1407.0184, accepé pour publication à Annali della Scuola Normale Superiore.

Le second chapitre se consacre aux quotients cubiques de l'algèbre du groupe de tresses à travers mes deux collaborations avec Ivan Marin:

- A cubic defining algebra for the Links-Gould polynomial, avec Ivan Marin, Adv. Math. 248 (2013), 1332-1365.
- Markov traces on the Birman-Wenzl-Murakami algebras, avec Ivan Marin, math/1403.4021.

Le dernier chapitre s'attache à présenter mes deux travaux les plus récents dans le domaine de la catégorification, celui avec Pedro Vaz s'intéressant à une catégorification de l'algèbre BMW et celui avec Agnès Gadbled et Anne-Laure Thiel sur une action catégorique du groupe de tresses affines étendu de type A:

- A remark on BMW algebra, q-Schur algebras and categorification, avec Pedro Vaz, Canad. J. Math. 66 (2014), no. 2, 453-480.
- Categorical action of the extended braid group of affine type A, avec Agnès Gadbled and Anne-Laure Thiel, math/1504.07596, accepté pour publication à Communications in Contemporary Mathematics.

L'appendice présente une formulation originale du polynôme d'Alexander obtenue à la suite de mon travail avec Jean-Marie Droz [11] et jamais publiée car à mon sens il manque une bonne application de cette formule. Nous avons néanmoins saisi l'opportunité donnée par cette HDR pour en laisser une trace.

Les thèmes très divers développés dans les trois chapitres ont en réalité une origine commune: tous les travaux proviennent d'une question à propos de la catégorification. Il n'y a aucune surprise concernant le dernier chapitre qui traite directement de questions relatives à cette direction de recherche. Nous retraçons maintenant l'historique de ces travaux.

Ma thèse de doctorat portait sur les homologies d'entrelacs de Khovanov et Rozansky [27] et la majorité des mes travaux qui ont suivi juste après restait dans ce domaine de recherche. J'étais essentiellement intéressé par les homologies de Khovanov et de Khovanov-Rozansky en lien avec l'homologie des nœuds de Heegaard-Floer espérant que ces catégorifications aideraient à comprendre la nature topologique des invariants quantiques, en particulier à travers la suite spectrale conjecturale entre l'homologie de Khovanov et l'homologie des nœuds de Heegard-Floer. Mon étude des homologies d'entrelacs de Khovanov et Rozansky a suggéré l'existence d'une invariant polynomial satisfaisant une relation d'écheveaux cubique. C'est de cette manière que j'ai commencé mon étude avec Ivan Marin des quotients cubiques de l'algèbre du groupe de tresses, en remarquant que l'on ne savait que très peu de choses à leurs sujets. Les quotients cubiques de l'algèbre du groupe de tresses nécessitent selon moi des recherches plus avancées car ils sont intéressants par plusieurs aspects. A peu près à la même époque j'ai commencé à réfléchir avec Pedro Vaz à la catégorification d'un quotient cubique particulier, l'algèbre BMW.

J'ai ensuite participé avec Benjamin Audoux, Paolo Bellingeri et Jean-Baptiste Meilhan au projet ANR JCJC Vaskho dont un des buts était l'étude des propriétés de type fini des homologies d'entrelacs. Notre travail commun commença suite à une rencontre à Dijon dont le but était d'introduire différentes définitions des invariants de Milnor pour voir si et comment on pouvait les catégorifier. Par exemple catégorifier l'enlacement d'une manière calculable est toujours une question ouverte. La stratégie générale est qu'il devait être plus facile d'étudier des propriétés de type fini sur des catégorifications d'invariants plus simples que directement sur les homologies d'entrelacs.

Nos discussions avec Michael Eisermann commencèrent après sa participation à une des rencontres de l'ANR Vaskho. Notre but était de comprendre comment étendre son critère de divisibilité du polynôme de Jones pour les entrelacs rubans à l'homologie de Khovanov. Lors d'une de nos discussions, nous réalisâmes que ces arguments de preuve étaient complètement locaux et nous divergèrent vers notre actuel projet commun. La question initiale est toujours largement ouverte.

La présentation dans les chapitres diffèrent quelque peu de celle dans les papiers originaux. Certains paragraphes ont été reproduits avec pas ou très peu de modifications. Certains papiers sont présentés dans le même ordre que l'original [37], certains dans un ordre complètement différent [36], certains sont exposés avec un éclairage particulier en privilégiant un aspect du papier original [2], [14], [53]... L'objectif recherché dans l'exposition de chaque chapitre est d'amener aussi naturellement que possible les recherches futures que je compte mener et qui sont présentées à la fin de chaque chapitre. Concernant justement ces sous-sections "Work in progress, open problems and perspective", elles présentent des projets qui sont à divers degrés d'avancement. Pour finir, j'ajouterai qu'il y a d'autres projets qui ne sont pas directement reliés aux résultats présentés dans ce manuscrit et qui me ramèneront encore plus vers le domaine de la catégorification et justifieront encore plus le titre de cette HDR.

Foreword

An HDR is supposed usually to present works of the author done after the Phd, this is obviously the case of this manuscript too, but also usually gives the author's point of view on a subject and contains a personal survey of an area of research, this is less the case and instead we present various facets of an area of research. In a broad sense, my area of research is known as *Q*uantum Topology, but not in a very precise direction. My objects of studies are of topological nature, braids, knots and surfaces and the techniques of various algebraic natures: algebraic topology, combinatorics, representation theory and categorical constructions. The works presented in this manuscript reflect various interplays between topology and algebra and you will find below a list of my papers, and see that this HDR focuses only on part of them. It is divided in three chapters, each of them focusing on a different thematic and the choice of papers follows from the choice of these thematics. Each of the chapter reflects one more precise direction which I believe deserve a particular attention.

The first chapter in some sense studies surfaces in the 4-space on their own, in connection with quantum and classical link invariants. It reports on one hand on my joint work in progress with Michael Eisermann and on the other hand on my joint work with Benjamin Audoux, Paolo Bellingeri and Jean-Baptiste Meilhan:

- Slice links and the Jones polynomial, with Michael Eisermann, in preparation.
- Homotopy classification of ribbon tubes and welded string links, with Benjamin Audoux, Paolo Bellingeri and Jean-Baptiste Meilhan, math/1407.0184, accepted for publication in Annali della Scuola Normale Superiore.

The second chapter is dedicated to cubical quotients of the braid group algebra and presents my joint works with Ivan Marin:

- A cubic defining algebra for the Links-Gould polynomial, with Ivan Marin, Adv. Math. 248 (2013), 1332-1365.
- Markov traces on the Birman-Wenzl-Murakami algebras, with Ivan Marin, math/1403.4021.

The last chapter deals with categorifications and discuss my joint work with Pedro Vaz on a categorification of the BMW algebra and my joint work with Agnès Gadbled and Anne-Laure Thiel on a categorical action of the extended affine type A braid group:

- A remark on BMW algebra, q-Schur algebras and categorification, with Pedro Vaz, Canad. J. Math. 66 (2014), no. 2, 453-480.
- Categorical action of the extended braid group of affine type A, with Agnès Gadbled and Anne-Laure Thiel, math/1504.07596, accepted for publication in Communications in Contemporary Mathematics.

The appendix presents an original description of the Alexander polynomial which was obtained as a sequel to the joint work with Jean-Marie Droz [11] and was never published because I had no good applications of this formula. We take the chance to include it in this manuscript to keep track.

The not completely related topics of the three chapters have in fact something in common: all the works origin in a question about categorification. This is no surprise for the last chapter since in this case the results deal with categorification. Let me recall here the history of these works.

My PhD thesis was about the link homology theory constructed by Khovanov-Rozansky [27] and most of the work I have done shortly afterwards was in this area of research. I was mostly interested in Khovanov and Khovanov-Rozansky link homologies with a view toward knot Heegaard-Floer homology hoping that these categorifications could help to understand the topological nature of the quantum invariants, in particular through the conjectural spectral sequence between Khovanov homology and knot Heegaard-Floer homology. My studies of the Khovanov-Rozansky link homologies suggested the existence of a polynomial invariant satisfying a cubical skein relation. This is how I started with Ivan Marin our studies of cubical quotients of the braid group algebra, noticing that not much was known about them. The cubical quotients of the braid group algebra really deserve further investigations, and are interesting from various perspectives. At the same time, I started to think with Pedro Vaz about categorification of a particular cubical quotient of the braid group algebra, namely the BMW algebra.

Afterwards I was involved in the JCJC ANR project Vaskho with Benjamin Audoux, Paolo Bellingeri and Jean-Bapstiste Meilhan. One of its goals was to investigate connections between finite type properties and link homology. The joint project started during one of our meetings in Dijon whose aim was to introduce various definitions of the Milnor invariants to see if/how one could categorify them. For instance, categorifying linking numbers in a computable fashion is still an open question. The general strategy was that it would be easier to prospect finite type properties on categorification of simpler invariants than on the link homology directly.

We started discussing with Michael Eisermann after he participated in a joint meeting of the ANR Vaskho. Our goal was to understand how the divisibility property of the Jones polynomial for ribbon links reflects in the Khovanov homology. During one of our discussions we realized his arguments of proof were completely local and we diverged to the present common work. The initial question is still completely open.

Let me add here a few words about the content of the chapters compared to the original papers. Certain paragraphs are reproduced from the papers with little or no changes. Some papers are presented here in the original order [37], some are in a completely different order [36], some others have a particular exposition that will shed more light on one aspect of the original paper [2], [14], [53]... The general goal of the exposition inside each chapter was to present the papers with a view toward the further developments I plan to investigate and which are presented at the end of each chapter. Concerning these subsections "Work in progress, open problems and perspective" they present projects which are at very different stages of investigations. Let me add also that some projects are not developed in this manuscript because not directly connected to the results presented here and will bring me even more back into the realm of categorification and hopefully will explain even more the title of this HDR.

List of works.

Papers extracted from the PhD thesis:

- Khovanov-Rozansky Graph Homology and Composition Product, Journal of Knot Theory and its Ramifications, 17, 12 (2008).
- Khovanov-Rozansky homology for embedded graphs, Fundamenta Mathematicae 214 (2011), 201-214.

Papers not presented in this manuscript and not directly connected to it:

- Grid diagrams and Khovanov homology, with Jean-Marie Droz, Algebraic and Geometric Topology, Volume 9, issue 3 (2009).
- On link homology theories and extended cobordism, with Anna Beliakova, Quantum Topol. 1 (2010), no. 4, 379-398.
- The homology of digraphs as a generalisation of Hochschild homology, with Paul Turner, J. Algebra Appl. 11 (2012), no. 2, 1250031, 13 pp.
- HOMFLY-PT skein module of singular links in the three-sphere, with Luis Paris, J. Knot Theory Ramifications 22 (2013), no. 2, 1350005 (13 pp).

Papers mentioned in this manuscript and not developed:

- On Usual, Virtual and Welded knotted objects up to homotopy, with Benjamin Audoux, Paolo Bellingeri and Jean-Baptiste Meilhan, math/1507.00202, accepted for publication in Journal of the Mathematical Society of Japan.
- On forbidden moves and the Delta move, with Benjamin Audoux, Paolo Bellingeri and Jean-Baptiste Meilhan, math/1510.04237.
- The HOMFLYPT polynomials of sublinks and the Yokonuma-Hecke algebras, with Loic Poulain d'Andecy, math/1606.00237.

Papers developed in this manuscript:

- A remark on BMW algebra, q-Schur algebras and categorification, with Pedro Vaz, Canad. J. Math. 66 (2014), no. 2, 453-480.
- A cubic defining algebra for the Links-Gould polynomial, with Ivan Marin, Adv. Math. 248 (2013), 1332-1365.
- Markov traces on the Birman-Wenzl-Murakami algebras, with Ivan Marin, math/1403.4021.
- Homotopy classification of ribbon tubes and welded string links, with Benjamin Audoux, Paolo Bellingeri et Jean-Baptiste Meilhan, math/1407.0184, accepted for publication at Annali della Scuola Normale Superiore.
- Categorical action of the extended braid group of affine type A, with Agnès Gadbled and Anne-Laure Thiel, math/1504.07596, accepted for publication at Communications in Contemporary Mathematics.
- Slice links and the Jones polynomial, with Michael Eisermann, in preparation.

Contents

Chapter 1

Surfaces in four-space: from links to link-homotopy

1.1 Sliceness obstructions from the Jones polynomial.

In a joint work with Eisermann, we extend his previous result on ribbon links and the Jones polynomial to the case of slice links. Along the way, we prove an extension of his result to a tangle setting and prove a link version of the famous result of Fox stating that slice knots are stably ribbon, i.e. for each slice knot there exists a ribbon knot such that the connected sum of the two is ribbon.

Recall that a link with n components is called ribbon if it bounds n immersed disks with only ribbon singularities. We describe in Figure 1.1 the local model of this type of singularities. A link with n components in the three sphere is said to be slice if it bounds n disjoint smooth disks in the four ball. Recall also that ribbon implies slice. The converse is an open problem known as the Slice-Ribbon conjecture

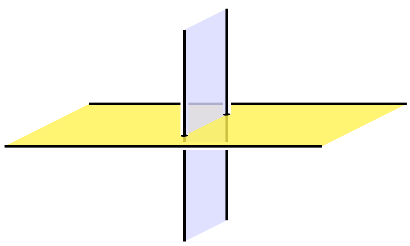


Figure 1.1: Local model for a ribbon singularity

We obtain the following theorem:

Theorem 1.1. *For every slice link $L \subset S^3$ with n components, the unnormalized Jones polynomial $V(L) \in \mathbb{Z}[q^{\pm 1}]$ is divisible by the Jones polynomial $V(\bigcirc^n) = (q + q^{-1})^n$ of the trivial link.*

Eisermann established the theorem for ribbons links, hence, this result is from some point of view disappointing since it indicates that one cannot use this criterion to find a counterexample to the Slice-Ribbon conjecture. From another point of view it is one of the first interplay between classical quantum invariants and topology. Moreover the proof indicates a strategy to find a criterion which will obstruct sliceness but not necessary ribboness. We will expand on this at the end of the section.

The proof of this theorem has two main ingredients: firstly we extend the result of Eisermann to the case of ribbon tangles and secondly we use an extension of the slice is stably ribbon criterion to the almost pure tangle case and hence to the link case.

We obtain in particular the following result:

Theorem 1.2. *Consider an oriented link $L = L_1 \cup \cdots \cup L_n$ in $\mathbb{R}^3$ with numbered components. If L is slice then there exists a ribbon link $L' = L'_1 \cup \cdots \cup L'_n$ such that any connected sum $L\sharp L' = (L_1\sharp L'_1) \cup \cdots \cup (L_n \sharp L'_n)$ is ribbon.*

This theorem is a consequence of the next one. We say that a tangle $T \in \mathscr{T}(2n, 2n)$ with $2n$ endpoints at the bottom and $2n$ endpoints at the top is almost pure if it is a string link up to multiplication by a braid generator σ_{2i+1} with $i = 0, \dots, n-1$. We say that an almost pure tangle $T \in \mathscr{T}(2n, 2n)$ is ribbon if its plat closure $L = \mathrm{cl}(T)$ is a ribbon link and similarly it is slice if its plat closure $L = \mathrm{cl}(T)$ is a slice link. Notice that the fact that it is almost pure implies that its plat closure has exactly n components.

Theorem 1.3. *Let $T \in \mathscr{T}(2n, 2n)$ be an almost pure tangle. If T is slice, then there exists a almost pure ribbon tangle $T' \in \mathscr{T}(2n, 2n)$ such that their product $T \cdot T'$ is ribbon.*

A careful observation of the original proof of Eisermann's divisibility criterion for the Jones polynomial of ribbon links shows that it is completely local and can be immediately adapted to almost pure ribbon tangles. The slogan is then as follows: a sum of terms could be divisible whereas each individual term may not. We prove in fact that, giving a ribbon link with n components presented as a closure of an almost pure ribbon tangle, and expanding the tangle in the Kauffman bracket skein module and hence expressing it in the usual Temperley-Lieb basis, each individual term in this expression is divisible. This result also applies now to knots through the cabling procedure. Recall here that Eisermann's original divisibility criterion was obviously uneffective for knots and that in addition Habiro proved that the divisibility criterion was also satisfied by boundary links which prevents to use a cabling procedure to extend it to knots since cable links of knots are obviously boundary links. The problem is now computational to make it efficient for knots. We checked by computer using a three cable that the figure-eight knot is not ribbon, nor slice, but for this knot, the determinant already obstructs sliceness. We are looking now for better examples.

1.2 Ribbon tubes in the four-space.

In this section, we consider ribbon tubes and ribbon torus-links, which are natural 2-dimensional analogues of string links and links, respectively. We show how ribbon tubes naturally act on the reduced free group, and how this action classifies ribbon tubes up to link-homotopy. Let us mention here that we do not expand at all about the proofs and the techniques of proof used in this work. All the proofs rely on the interplay between topology and some combinatorial theories: welded knot theory and Gauss diagrams techniques. We choose to present here only the topological output.

First fix some notations. Let I be the unit closed interval. Let $[\![1, n]\!]$ be the set of integers between 1 and n. We fix n distinct points $\{p_i\}_{i\in[\![1,n]\!]}$ in I. For each $i \in [\![1, n]\!]$, we choose a disk D_i in the interior of the 2–ball $B^2 = I \times I$ which contains the point p_i, seen in $\{\frac{1}{2}\} \times I$, in its interior. We furthermore require that the disks D_i, for $i \in [\![1, n]\!]$, are pairwise disjoint. We denote by $C_i := \partial D_i$ the oriented boundary of D_i. We consider the 3–ball $B^3 := B^2 \times I$ and the 4–ball $B^4 = B^3 \times I$. For m a positive integer and for each submanifold $X \subset B^m \cong B^{m-1} \times I$, we set the notation

- $\partial_0 X = X \cap (B^{m-1} \times \{0\})$;
- $\partial_1 X = X \cap (B^{m-1} \times \{1\})$;
- $\partial_* X = \partial X \setminus (\partial_0 X \sqcup \partial_1 X)$;
- $\overset{*}{X} = X \setminus (\partial_* X \cup \partial(\partial_0 X) \cup \partial(\partial_1 X))$.

By a tubular neighborhood of X, we will mean an open set N such that $N \cap \mathring{B}^m$ is a tubular neighborhood of $\mathring{X}$ in $\mathring{B}^m$ and $\partial_\varepsilon N$ is a tubular neighborhood of $\partial_\varepsilon X$ in $\partial_\varepsilon B^m$ for both $\varepsilon = 0$ and 1.

1.2.1 Ribbon tubes and their homology

Definitions

Definition 1.1. *A* ribbon tube *is a locally flat embedding* $T = \bigsqcup_{i\in[\![1,n]\!]} A_i$ *of n disjoint copies of the oriented annulus* $S^1 \times I$ *in* $\overset{*}{B}{}^4$ *such that*

- $\partial A_i = C_i \times \{0,1\}$ *for all* $i \in [\![1,n]\!]$ *and the orientation induced by* A_i *on* ∂A_i *coincides with the one of* C_i*;*
- *there exist n locally flat immersed* 3*–balls* $\bigcup_{i\in[\![1,n]\!]} B_i$ *such that*
  - $\partial_* B_i = \mathring{A}_i$ *for all* $i \in [\![1,n]\!]$*;*
  - $\partial_\varepsilon B_i = D_i \times \{\varepsilon\}$ *for all* $i \in [\![1,n]\!]$ *and* $\varepsilon \in \{0,1\}$*;*
  - *the singular set of* $\bigcup_{i=1}^{n} B_i$ *is a disjoint union of so-called* ribbon singularities*, ie flatly transverse disks whose preimages are two disks, one in* $\bigcup_{i=1}^{n} \mathring{B}_i$ *and the other with interior in* $\bigcup_{i=1}^{n} \mathring{B}_i$*, and with boundary essentially embedded in* $\bigsqcup_{i=1}^{n} \partial_* B_i = \bigsqcup_{i=1}^{n} \mathring{A}_i$.

We denote by rT_n the set of ribbon tubes up to isotopy fixing the boundary circles. It is naturally endowed with a monoidal structure by the stacking product $T \bullet T' := T \bigcup_{\partial_1 T = \partial_0 T'} T'$, and reparametrisation, and with unit element the *trivial ribbon tube* $\mathbf{1}_n := \bigsqcup_{i\in[\![1,n]\!]} C_i \times I$.

Note that this notion of ribbon singularity is a 4–dimensional analogue of the classical notion of ribbon singularity introduced by R. Fox in [13] and that we discussed in the previous subsection. Similar ribbon knotted objects were studied, for instance, in [55], [56] and [23], and a survey can be found in [51].

Remark 1.1. *There are two natural ways to close a ribbon tube* $T \in rT_n$ *into a closed (ribbon) knotted surface in the* 4*–space. First, by gluing the disks* $\bigsqcup_{i\in[\![1,n]\!]} D_i \times \{0,1\}$ *which bound* $\partial_0 T$ *and* $\partial_1 T$*, and gluing a* 4*–ball along the boundary of* B^4*, one obtains a n-component ribbon* 2*–link [54], which we shall call the* disk-closure *of* T*. Second, by gluing a copy of the trivial ribbon tube* $\mathbf{1}_n$ *along* T*, identifying the pair* $(B^3\times\{0\}, \partial_0 T)$ *with* $(B^3\times\{1\}, \partial_1 \mathbf{1}_n)$ *and* $(B^3\times\{1\}, \partial_1 T)$ *with* $(B^3\times\{0\}, \partial_0 \mathbf{1}_n)$*, and taking a standard embedding of the resulting* $S^3 \times S^1$ *in* S^4*, one obtains a n-component ribbon torus–link [46], which we shall call the* tube-closure *of* T*. This is a higher dimensional analogue of the usual braid closure operation.*

An element of rT_n is said to be *monotone* if it has a representative which is flatly transverse to the lamination $\bigcup_{t\in I} B^3 \times \{t\}$ of B^4.
We denote by rP_n the subset of rT_n whose elements are monotone.

Proposition 1.1. *The set* rP_n *is a group for the stacking product.*

Homology groups

Let T be a ribbon tube with tube components $\bigsqcup_{i\in[\![1,n]\!]} A_i$.

Since T is locally flat in B^4, there is a unique way, up to isotopy, to consider, for all $i \in [\![1,n]\!]$, disjoint tubular neighborhoods $N(A_i) \cong D^2 \times S^1 \times I$ for A_i, with $A_i = \{0\} \times S^1 \times I \subset N(A_i)$. We denote by $N(T) := \bigsqcup_{i\in[\![1,n]\!]} N(A_i)$ a reunion of such tubular neighborhoods and by $W = B^4 \setminus \overset{*}{N}(T)$ the complement of its interior in B^4.

For each $i \in [\![1,n]\!]$,

- the i^{th} *homological meridian* c_i of T is the homology class in $H_1(W)$ of $\partial D^2 \times \{s\} \times \{t\} \subset \partial N(A_i)$ for any $(s,t) \in S^1 \times I$;

- the i^{th} *homological meridional torus* τ_i of T is the homology class in $H_2(W)$ of $\partial D^2 \times S^1 \times \{t\} \subset \partial N(A_i)$ for any $t \in I$.

As a direct application of the Mayer–Vietoris exact sequence, we obtain:

Proposition 1.2. *The homology groups of W are* $H_0(W) = \mathbb{Z}$, $H_1(W) = \mathbb{Z}^n = \mathbb{Z}\langle c_i \mid i \in [\![1,n]\!]\rangle$, $H_2(W) = \mathbb{Z}^n = \mathbb{Z}\langle \tau_i \mid i \in [\![1,n]\!]\rangle$, $H_3(W) = \mathbb{Z}$ *and* $H_k(W) = 0$ *for* $k \geq 4$.

1.2.2 Broken surface diagrams and fundamental group

Links in 3–space can be described using diagrams, which are their generic projection onto a 2–dimensional plane with extra decoration encoding the 3–dimensional information. Similarly, it turns out that ribbon knotted objects, which are surfaces in 4–space, can be described using their generic projection onto a 3–space; this leads to the following notion of broken surface diagram.

Broken surface diagrams

Definition 1.2. *A* broken surface diagram *is a locally flat immersion S of n oriented annuli* $\bigsqcup_{i\in[\![1,n]\!]} A_i$ *in* $\overset{*}{B}{}^3$ *such that*

- $\partial A_i = C_i \times \{0,1\}$ *for all* $i \in [\![1,n]\!]$ *and the orientation induced by* A_i *on* ∂A_i *coincides with that of* C_i*;*
- *the set* $\Sigma(S)$ *of connected components of singular points in S consists of flatly transverse circles in* $\overset{n}{\underset{i=1}{\cup}} \mathring{A}_i$.

Moreover, for each element of $\Sigma(S)$*, a local ordering is given on the two circle preimages. By convention, this ordering is specified on pictures by erasing a small neighborhood in* $\overset{n}{\underset{i=1}{\cup}} \mathring{A}_i$ *of the lowest preimage (see Figure 1.2). Note that this is the same convention which is used for usual knot diagrams.*

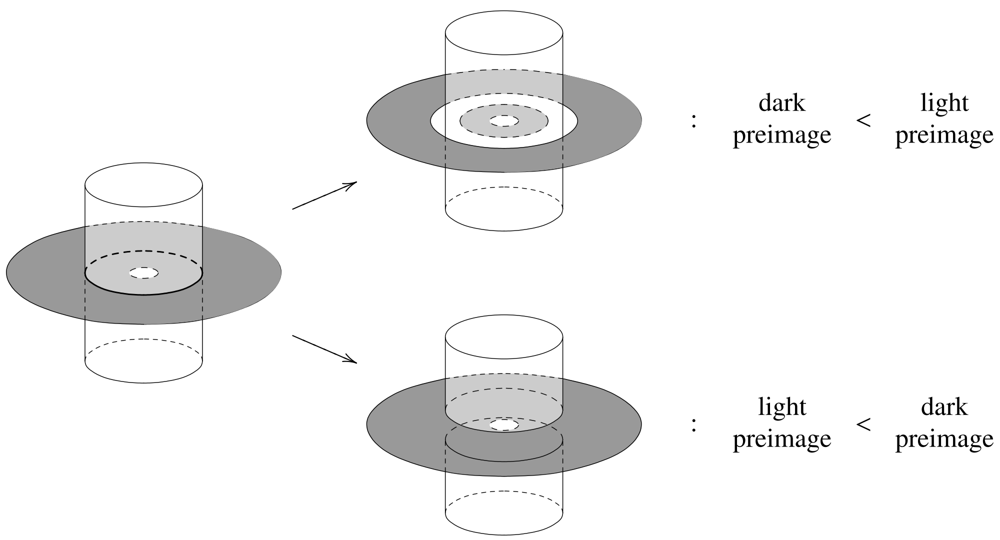


Figure 1.2: Local pictures for a singular circle in a broken surface diagram

Definition 1.3. *A broken surface diagram S is said to be* symmetric *if and only if,*

1. *for each of element of* $\Sigma(S)$*, one of the preimages is essential in* $\overset{n}{\underset{i=1}{\cup}} A_i$ *and the other is not;*

2. *there is a pairing $\Sigma(S) =: \sqcup_r \{c_1^r, c_2^r\}$ such that, for each r, the essential preimages of c_1^r and c_2^r*

   i. *are respectively lower and higher than their non essential counterparts;*

   ii. *bound an annulus in $\bigcup_{i=1}^{n} \mathring{A}_i$;*

   iii. *this annulus avoids $\Sigma(S)$.*

As a consequence of the remark below, a symmetric broken surface diagram looks locally like in Figure 1.3.

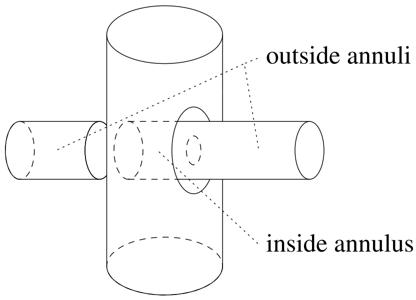


Figure 1.3: A local picture for paired singular circles in symmetric broken surface diagrams

Remark 1.2. *For a symmetric broken surface diagram S, the essential preimages of $\Sigma(S)$ cut the annuli of S into smaller annular pieces. Moreover, condition (1) above implies that there is a well defined notion of inside/outside for each annulus of S. Then, it follows from condition (2ii.) that the annular pieces between two paired essential preimages are exactly the portions of S which are inside S. Accordingly, we called these annular pieces* inside annuli*, and the other pieces* outside annuli *(see Figure 1.3). Condition (2iii.) implies furthermore that both boundary components of an inside annulus belongs to the same outside annulus.*

Let T be a ribbon tube, and consider a projection $B^4 \to \mathrm{B}^3$ which is generic with respect to T. Then the image of T in B^3 has singular locus a union of double points arranged in flatly transverse circles, and for each double point, the preimages are naturally ordered by their positions on the projection rays. This suggests that broken surface diagrams can be thought of as 3–dimensional representations of ribbon tubes. This is indeed the case, as stated in the next result, which is essentially due to Yanagawa.

Lemma 1.1. *[56] Any generic projection of a ribbon tube from B^4 into B^3 is a broken surface diagram. Conversely any broken surface diagram is the projection of a ribbon tube.*

More specifically, we have the following.

Lemma 1.2. *[55, 23] Any ribbon tube can be represented by a symmetric broken surface diagram.*

Fundamental groups and Wirtinger presentation

Let T be a ribbon tube with tube components $\bigsqcup_{i\in[\![1,n]\!]} A_i$ and define $N(T) := \bigsqcup_{i\in[\![1,n]\!]} N(A_i)$ and W as in the previous subsection. We also consider a global parametrization (x, y, z, t) of B^4, which is compatible with $B^4 \cong B^3 \times I \cong B^2 \times I \times I$ near $\partial_0 B^4$ and $\partial_1 B^4$, and such that the projection along z maps T onto a symmetric broken surface diagram S. We also fix a base point $e := (x_0, y_0, z_0, t_0)$ with z_0 greater than the highest z–value taken on $N(T)$.

Notation 1. *We set $\pi_1(T) := \pi_1(W)$ with base point e.*

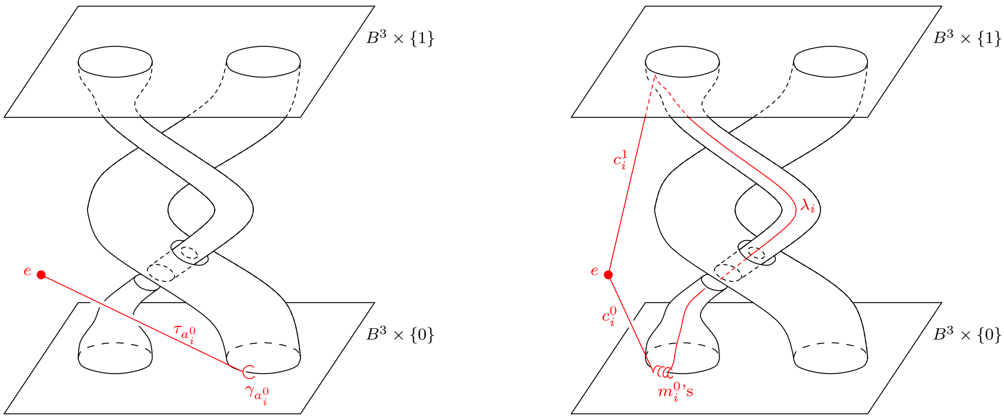
Elements of the fundamental group: For every point $a := (x_a, y_a, z_a, t_a) \in T$, we define $m_a \in \pi_1(T)$, the *meridian around a*, as $\tau_a^{-1}\gamma_a\tau_a$ where

- τ_a is the straight path from e to $\widetilde{a} := (x_a, y_a, z_0, t_a)$;
- γ_a is the loop in W, unique up to isotopy, based at $\widetilde{a}$ and which enlaces positively T around a.

In particular, we define:

Notation 2. *For each $i \in [\![1, n]\!]$ and $\varepsilon \in \{0, 1\}$, we denote by m_i^ε the meridian in $\pi_1(T)$ defined as $m_{a_i^\varepsilon}$ for any $a_i^\varepsilon \in C_i \times \{\varepsilon\}$. If $\varepsilon = 0$, we call it a* bottom meridian*, and if $\varepsilon = 1$, we call it a* top meridian*; see the left-hand side of Figure 1.4 for an example of a bottom meridian.*

Note that, for any $\varepsilon \in \{0, 1\}$ and any choice of a_i^ε, the fundamental group of $\partial_\varepsilon W$ based at $(x_0, y_0, z_0, \varepsilon)$ can be identified with the free group $\mathrm{F}_n = \langle m_i^\varepsilon \mid i \in [\![1, n]\!]\rangle$.



meridian m_i^0 longitude λ_i

Figure 1.4: Examples of meridians and longitude

Now, we define the notion of longitude for T as follows. First, we fix two points $e_i^0 \in \partial_0 N(A_i)$ and $e_i^1 \in \partial_1 N(A_i)$ on each extremity of the boundary of the tubular neighborhood of A_i. A longitude for A_i is defined as the isotopy class of an arc on $\partial N(A_i)$ running from e_i^0 to e_i^1, see the right-hand side of Figure 1.4 for an example. Since $N(A_i)$ is homeomorphic to $D^2 \times S^1 \times I$, we can note that

$$\partial N(A_i) = (S^1 \times S^1 \times I) \cup (D^2 \times S^1 \times \{0\}) \cup (D^2 \times S^1 \times \{1\}),$$

so that the choice of a longitude for A_i is *a priori* specified by two coordinates, one for each of the two S^1–factors in $S^1 \times S^1 \times I$. On one hand, the first S^1–factor is generated by the meridian m_i, so that the first coordinate is given by the linking number with the tube component A_i. It can be easily checked, on the other hand, that two choices of longitude for A_i which only differ by their coordinate in the second S^1–factor are actually isotopic in W.

Definition 1.4. *For each $i \in [\![1, n]\!]$, we call i^{th}* longitude of T *the isotopy class of an arc on $\partial N(A_i)$, running from e_i^0 to e_i^1, and closed into a loop with an arc $c_i^0 \cup c_i^1$ defined as follows. For $\varepsilon \in \{0, 1\}$, we denote by $\widetilde{e}_i^\varepsilon$ the point above e_i^ε with z–coordinate z_0; then c_i^ε is the broken line between e, $\widetilde{e}_i^\varepsilon$ and e_i^ε. See the right-hand side of Figure 1.4.*

In the following, we give a presentation for $\pi_1(T)$ in terms of broken surface diagrams.

Let S be a symmetric broken surface diagram representing T. According to the notation set in Remark 1.2, we denote by $\mathrm{Out}(S)$ the set of outside annuli of S and by $\mathrm{In}(S)$ the set of inside annuli. For each $\beta \in \mathrm{In}(S)$, we define

- $\alpha_\beta^0 \in \mathrm{Out}(S)$ the outside annulus which contains $\partial\beta$;
- C_β the connected component of $\beta \cap \alpha_\beta^0$ which is closer to $\partial_0 T$, according to the co–orientation order defined after definition 1.1;
- $\alpha_\beta^- \in \mathrm{Out}(S)$ the outside annulus which has C_β as a boundary component;
- $\alpha_\beta^+ \in \mathrm{Out}(S)$ the outside annulus which has $(\beta \cap \alpha_\beta^0) \setminus C_\beta$ as a boundary component;
- $\varepsilon_\beta = 1$ if, according to the local ordering, the preimage of C_β in β is higher than the preimage in α_β^0, and $\varepsilon_\beta = -1$ otherwise.

See Figure 1.5 for an example.

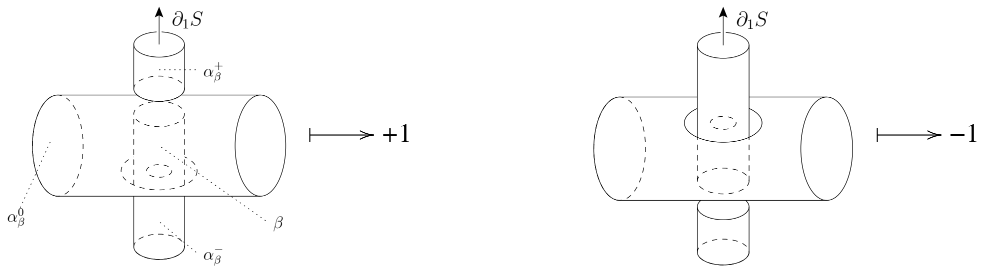


Figure 1.5: Signs associated to inside annuli

Proposition 1.3. *[54, 56] Let T be a ribbon tube and S any broken surface representing it, then*

$$\pi_1(T) \cong \left\langle Out(S) \;\middle|\; \alpha_\beta^+ = (\alpha_\beta^0)^{\varepsilon_\beta}\alpha_\beta^-(\alpha_\beta^0)^{-\varepsilon_\beta} \text{ for all } \beta \in In(S) \right\rangle.$$

In this isomorphism, $\alpha \in Out(S)$ is sent to m_a, where a is any point on α close to $\partial\alpha$.

Corollary 1.1. *The group $\pi_1(T)$ is generated by elements $\{m_a\}_{a\in T}$, and moreover, if $a \in A_i$ for some $i \in [\![1,n]\!]$, then m_a is a conjugate of m_i^ε for both $\varepsilon = 0$ or 1.*

Reduced fundamental group

In this section, we define and describe a reduced notion of fundamental group for T. Indeed, Corollary 1.1 states that $\pi_1(T)$ is normally generated by meridians $m_1^\varepsilon, \cdots, m_n^\varepsilon$ for either $\varepsilon = 0$ or 1. Moreover, since top meridians are also conjugates of the bottom meridians and *vice versa*, we can define the following without ambiguity:

Definition 1.5. *The* reduced fundamental group $R\pi_1(T)$ of T *is defined as the smallest quotient of $\pi_1(T)$ where each bottom (or top) meridian commutes with all its conjugates.*
For convenience , we also denote $R\pi_1(\partial_\varepsilon W)$ by $R\pi_1(\partial_\varepsilon T)$, for $\varepsilon \in \{0,1\}$.

It is a consequence of the description of $H_*(W)$ given in Proposition 1.2 that, for $\varepsilon \in \{0,1\}$, the inclusion $\iota_\varepsilon : \partial_\varepsilon W \hookrightarrow W$ induces isomorphisms at the H_1 and H_2 levels. Stallings theorem, *i.e.* theorem 5.1 in [50], then implies that

$$(\iota_\varepsilon)_k : \pi_1(\partial_\varepsilon W)/\Gamma_k\pi_1(\partial_\varepsilon W) \xrightarrow{\simeq} \pi_1(T)/\Gamma_k\pi_1(T)$$

are isomorphisms for every $k \in \mathbb{N}^*$. But $\pi_1(\partial_\varepsilon W)$ is the free group F_n generated by meridians $m_1^\varepsilon, \cdots, m_n^\varepsilon$. It follows from Habegger-Lin's Lemma 1.3 in [15] that for $k \geq n$, $R\left(\mathrm{F}_n/\Gamma_k\mathrm{F}_n\right) \cong \mathrm{RF}_n$. As a consequence:

Proposition 1.4. *The inclusions ι_0 and ι_1 induce isomorphisms*

$$RF_n \cong R\pi_1(\partial_0 T) \xrightarrow[\iota_0^*]{\simeq} R\pi_1(T) \xleftarrow[\iota_1^*]{\simeq} R\pi_1(\partial_1 T) \cong RF_n.$$

Using the isomorphisms of Proposition 1.4, we define, for every ribbon tube T, a map $\varphi_T : \mathrm{RF}_n \to \mathrm{RF}_n$ by $\varphi_T := \iota_0^{*-1} \circ \iota_1^*$. This can be seen as reading the top meridians as products of the bottom ones. It is straightforwardly checked that $\varphi_{T \bullet T'} = \varphi_T \circ \varphi_{T'}$ and this action on RF_n is obviously invariant under isotopies of ribbon tubes.

It follows from Corollary 1.1 that:

Proposition 1.5. *For every ribbon tube T, φ_T is an element of $Aut_C(RF_n)$, the group of conjugating automorphisms which send each bottom (or top) meridian to a conjugate of bottom (or top) meridian. More precisely, the action of $T \in rT_n$ on RF_n is given by conjugation of each generator x_i, for $i \in [\![1, n]\!]$, by the image through ι_0^* of an i^{th} longitude of T.*

Note that, in the reduced free group, this conjugation does not depend on the choice of a i^{th} longitude.

1.2.3 Classification results

Link-homotopy

Definition 1.6. *A* singular ribbon tube *is a locally flat immersion T of n annuli $\bigsqcup_{i \in [\![1,n]\!]} A_i$ in $\mathring{B}^4$ such that*

- *$\partial A_i = C_i \times \{0, 1\}$ for all $i \in [\![1, n]\!]$ and the orientation induced by A_i on ∂A_i coincides with that of C_i;*
- *the singular set of T is a single flatly transverse circle, called* singular loop*, whose preimages are two circles embedded in $\bigcup_{i=1}^{n} \mathring{A}_i$, an essential and a non essential one.*
- *there exist n locally flat immersed 3–balls $\bigcup_{i \in [\![1,n]\!]} B_i$ such that*
  - *$\partial_* B_i = \mathring{A}_i$ and $\partial_\varepsilon B_i = D_i \times \{\varepsilon\}$ for all $i \in [\![1, n]\!]$ and $\varepsilon \in \{0, 1\}$;*
  - *the singular set of $\bigcup_{i=1}^{n} B_i$ is a disjoint union of flatly transverse disks, all of them being ribbon singularities but one, whose preimages are two disks bounded by the preimages of the singular loop, one in $\bigcup_{i=1}^{n} \partial_* B_i$ and the other with interior in $\bigcup_{i=1}^{n} \mathring{B}_i$.*

We say that a singular ribbon tube is self-singular *if and only if both preimages of the singular loop belong to the same tube component.*

Definition 1.7. *Two ribbon tubes T_1 and T_2 are said to be* (link-)homotopic *if and only if there is a 1–parameter family of regular and (self-)singular ribbon tubes from T_1 to T_2 passing through a finite number of (self-)singular ribbon tubes.*
We denote by rT_n^h the quotient of rT_n by the link-homotopy equivalence, which is compatible with the monoidal structure of rT_n. Furthermore, we denote by rP_n^h the image of rP_n in rT_n^h.

Proposition 1.6. *[23] The homotopy equivalence is generated by* circle crossing changes*, which are the operations in B^4 induced by the local move shown in Figure 1.6, which switches the local ordering on the preimages of a given singular circle.*
The link-homotopy equivalence is generated by self-circle crossing changes*, where it is furthermore required that both preimages are on the same tube component.*

Note that a circle crossing change can be seen as a local move among symmetric broken surface diagrams. Indeed, although applying a circle crossing change yields a surface diagram which is no longer symmetric (see the middle of Figure 1.7), the resulting "paired essential preimages with same ordering" corresponds to a piece of tube passing entirely above or below another piece of tube. There is thus no obstruction in B^4 for pushing these two pieces of tube apart, so that their projections don't meet anymore (see the right-hand side of Figure 1.7).

We now state one of the main results of this subsection.

Theorem 1.4. *Every ribbon tube is link-homotopic to a monotone ribbon tube.*

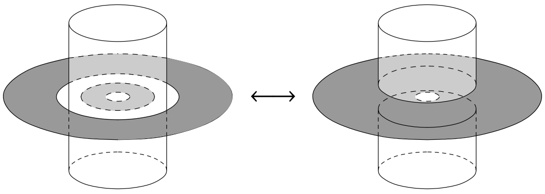


Figure 1.6: A circle crossing change at the level of broken surface diagrams

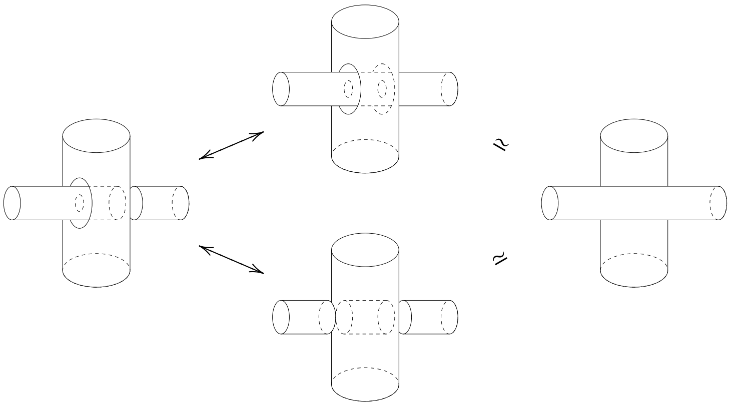


Figure 1.7: Circle crossing change at the symmetric broken surface diagram level

As mentioned before the proof of this theorem is done completely in the realm of Gauss diagrams and the topological interpretation follows from the connections between welded knot theory and ribbon tori through the Tube map introduced by Yajima in the classical case and extended to welded links by Satoh.

Action on the reduced free group and link-homotopy

Previously, a conjugating automorphism φ_T was associated to any ribbon tube T. It turns out that this automorphism φ_T is invariant under link-homotopy.

Proposition 1.7. *If T_0 and T_1 are two link-homotopic ribbon tubes, then $\varphi_T = \varphi_{T'}$.*

We can now give the main result of this section

Theorem 1.5. *The map $\varphi : rT_n^h \longrightarrow Aut_C(RF_n)$, sending T to φ_T is an isomorphism.*

One can reformulate the theorem using Milnor invariants which can be seen as numerical invariants encoding the associated automorphism, see the last section of [2]. This gives also a natural context for an extension of Milnor invariants to welded links.

Classification of ribbon torus links up to link-homotopy

In [16] a structure theorem was given for certain "concordance-type" equivalence relations on links. Here we give an analogous structure theorem in the higher dimensional case. Actually, we follow the reformulation given in [17], which was in fact implicit in the proof of [16].

We consider n-component *ribbon torus-links*, that is, locally flat embeddings of n disjoint tori in S^4 which bound locally flat immersed solid tori whose singular set is a finite number of ribbon disks. Denote by rL_n the set of n-component ribbon torus-links up to isotopy. The tube-closure operation defined in

Remark 1.1 induces a natural closure map $\hat{}$: $\mathrm{rT}_n \to \mathrm{rL}_n$, which is easily seen to be surjective. Indeed, given an n-component ribbon torus-link, it is always possible up to isotopy to find a 3-ball intersecting the n components transversally exactly once, along an essential circle, so that cutting the ribbon torus-link along this ball provides a preimage for the closure operation. We shall refer to such a ball as a "base-ball".

Consider an equivalence relation E on the union, for all $n \in \mathbb{N}^*$, of the sets rT_n and rL_n. We will denote by $E(x)$ the E-equivalence class of a ribbon tube or torus-link x, and we also denote by E the map which sends a ribbon knotted object to its equivalence class. We denote respectively by $E\mathrm{rT}_n$ and $E\mathrm{rL}_n$ the set of E-equivalence classes of ribbon tubes and ribbon torus-links.

The Habegger-Lin Classification Scheme relies on the following set of axioms:

(1) The equivalence relation E is *local*, i.e. for all $L_1, L_2 \in \mathrm{rT}_n$ such that $E(L_1) = E(L_2)$, and for all $T_1, T_2 \in \mathrm{rT}_{2n}$ such that $E(T_1) = E(T_2)$, we have

 (i) $E(\hat{L}_1) = E(\hat{L}_2)$,

 (ii) $E(\mathbf{1}_n \otimes L_1) = E(\mathbf{1}_n \otimes L_2)$, where $\otimes$ denotes the horizontal juxtaposition,

 (iii) $E(L_1 \lhd T_1) = E(L_2 \lhd T_2)$ and $E(T_1 \rhd L_1) = E(T_2 \rhd L_2)$, where the left action $\lhd$ and right action $\rhd$ of rT_{2n} on rT_n are defined in Figure 1.8.

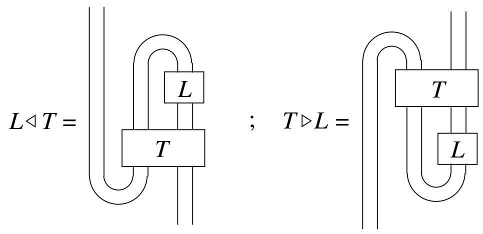


Figure 1.8: Shematical representations of the left and right actions of $T \in \mathrm{rT}_{2n}$ on $L \in \mathrm{rT}_n$

(2) For all $L \in \mathrm{rT}_n$, there is a string link L', such that $E(L \cdot L') = E(\mathbf{1}_n)$.

(2′) For all $L \in \mathrm{rT}_n$, $E(L \cdot \overline{L}) = E(\mathbf{1}_n)$, where $\overline{L}$ denotes the image of L under the hyperplane reflexion about $B^3 \times \{\frac{1}{2}\}$.

(3) The equivalence relation E on ribbon torus-links is generated by isotopy of ribbon torus-links and the equivalence relation E on ribbon tubes: if L and L' are two ribbon torus-links such that $E(L) = E(L')$, then there is a finite sequence $L_1, \dots, L_m$ of ribbon tubes such that L is isotopic to $\hat{L}_1$, L' is isotopic to $\hat{L}_m$, and for all i $(1 \leq i < m)$, either $E(L_i) = E(L_{i+1})$ or $\hat{L}_i$ is isotopic to $\hat{L}_{i+1}$.

Let E be a local equivalence relation. Denote respectively by ES_n^R and ES_n^L the right and left stabilizers of the trivial ribbon tube in $E\mathrm{rT}_n$. One can easily check that ES_n^R and ES_n^L are both submonoids of rT_{2n}. Furthermore, the closure operation induces a map $\hat{}$: $E\mathrm{rT}_n \to E\mathrm{rL}_n$ which passes to the quotient by ES_n^R (resp. ES_n^L).

Now, assume in addition that the equivalence relation E satisfies Axiom (2). Then clearly the monoid $E\mathrm{rT}_n$ is a group, and both ES_n^R and ES_n^L are subgroups of $E\mathrm{rT}_{2n}$. If the stronger Axiom (2′) holds, then we actually have $ES_n^R = ES_n^L$.

Theorem 1.6 (Structure theorem for ribbon torus-links)**.**

- *Let E be a local equivalence relation satisfying Axiom* (2)*. Then, for* $* = R$ *or L, the quotient map*

$$rT_n \longrightarrow ErT_n/ES_n^*$$

factors through the closure map, i.e., we have a ribbon torus-link invariant

$$\widetilde{E} : rL_n \longrightarrow ErT_n/ES_n^*$$

such that the composite map to ErL_n *is E.*

- *Furthermore, if Axiom* (3) *also holds, then we have a bijection*

$$ErT_n/ES_n^* = ErL_n.$$

This structure theorem is shown by applying verbatim the arguments of [16], as reformulated in Theorem 3.2 of [17]. Indeed, although these papers only deal with classical knotted objects, the proof is purely combinatorial and algebraic, and involves no topological argument *except* for [16, Prop. 2.1], whose ribbon tube analogue can actually be shown by a straightforward adaptation of Habegger and Lin's arguments.

We have the following classification result for ribbon torus-links up to link-homotopy.

Proposition 1.8. *The link-homotopy relation on ribbon tubes satisfies Axioms* (1)*,* (2′) *and* (3) *above. Consequently, we have a bijection*

$$rT_n^h/S_n^+ = rL_n^h,$$

where rL_n^h is the set of link-homotopy classes of ribbon torus-links and S_n^+ denotes the stabilizer of the trivial ribbon tube in rT_n^h with respect to the right (or left) action of rT_{2n} on rT_n defined in Figure 1.8.

1.3 Work in progress, open questions and perspectives

1.3.1 Sliceness and quantum invariants

In [12], Eisermann proposed to generalize his divisibility criterion for ribbon links from the Jones polynomial to other quantum invariants, in particular the HOMFLYPT polynomial. His suggestion was to continue to look at the divisibility of the invariant by the value of the unknot. It turned out later that Kazuo Habiro gave a counter-example showing that it was the wrong generalization of the criterion. It was already noticed by Eisermann in [12] that it was not working for the two-variable Kauffman polynomial. Recall here that one of the motivations and explanations for Eisermann to look to this precise value of the Jones polynomial was that in fact it is a value which gives a common evaluation of the Jones polynomial and the Alexander polynomial. Moreover they specialize to another classical invariant, namely the determinant which was known to be zero for ribbon links. Hence it is immediate that the Jones polynomial should be divisible at least once by the minimal polynomial of this value which is in fact the value of the Jones polynomial on the unknot. We conjecture the following generalization of Eisermann's criterion:

Conjecture 1.1. *Given a one variable polynomial link invariant P, a complex number a and a natural number k such that $P(a)(L) = det(L)^k$ for any link L, then for each ribbon link L with n components the polynomial $P(L)$ is divisible by the (nk)-th power of the minimal polynomial of a.*

We plan to start to investigate this conjecture in two particular cases: the first is the specialization of the HOMFLYPT polynomial corresponding to quantum $\mathfrak{sl}_{2n}$ (in this case $k = 1$) and the second is a common specialization of the two variables Kauffman polynomial and the two variable Links-Gould invariant (in this case $k = 2$). We already checked the divisibility criterion on the first prime links using the data provided by the LinkInfo webpage developed by Cha and Livingston.

One more long term project is to consider properties satisfied by ribbon links that cannot be transferred to slice links through the previous strategy of proof. One idea is to consider "positivity properties", study for instance the image of almost pure ribbon tangles in the Temperley-Lieb algebra from this prospective. We started to think about this with Eisermann.

1.3.2 Link-homotopy classification for general annuli

The work initiated in [2] with Audoux, Bellingeri and Meilhan already had further developments in the direction of the interactions between topology of ribbon surfaces on one hand and welded and virtual knot theory on the other hand which we do not present in this manuscript [4], [3]. Instead with present here a work in progress with Audoux and Meilhan aimed at generalizing the main results of [2] by removing the ribbon hypothesis. In particular, we conjecture the following :

Conjecture 1.2. *The statement of Theorem 1.5 remains true without the ribbon hypothesis on the annuli.*

We discuss now our strategy for the proof of this conjecture. The link-homotopy classification of spheres was settled by Bartels and Teichner [5]. They proved in two main steps that they are all link-homotopic to the trivial. First they showed that embedded spheres are always singular concordant to the trivial ones and then they promote this concordance to a link-homotopy. They have this result for all codimension 2 embeddings of sphere of dimension $n \geq 2$, but their proof simplifies in the case $n = 2$ because they understand completely the singular link concordance in terms of elementary ones. Using their proof we want to prove that annuli are always link-homotopic to ribbon ones. Then the conjecture will follow from our classification for ribbon ones and the fact that the morphism to the automorphism of the reduced free group passes to the quotient by the link homotopy relation for general annuli. Remark also that the fact that annuli are link-homotopic to ribbon ones immediately implies the result of Bartels-Teichner; hence it would be better to have a proof of the conjecture without relying on their results, but for the moment it is not clear how to do such a proof.

Chapter 2

Cubical quotients of the braid group and link invariants

2.1 On the Links-Gould invariant

2.1.1 Introduction and context

In 1992, Links and Gould introduced a polynomial invariant of knots and links out of a family of 4-dimensional representations of the quantum Lie superalgebra $\mathcal{U}_q(2|1)$. This invariant satisfies a *cubic* skein relation, that is the simplest skein relation that can be asked for, the quadratic one being characteristic of the HOMFLY-PT polynomial. It shares this property with the Kauffman polynomial (see next subsection) (which corresponds to the quantum orthosymplectic Lie algebras and their standard representations), but behaves quite differently, notably with respect to disjoint union of links : the Kauffman polynomial is multiplicative with respect to the disjoint union of two links, whereas the Links-Gould polynomial vanishes on such disjoint unions.

The additional skein relations satisfied by the Kauffman polynomial are relations of the so-called Birman-Wenzl-Murakami (BMW) algebra (see [6], [41]). This BMW_n algebra is a quotient of the group algebra KB_n of the braid group over some field K of characteristic 0 by a generic cubic relation and by some other relations in KB_3, and there exists a single Markov trace on the tower of algebras BMW_n, whose value on closed braids provides the Kauffman invariant. This algebra is a deformation of the classical algebra of Brauer diagrams, and as such admits a basis with a nice combinatorial description. It describes the centralizer algebra of the action of $\mathcal{U}_q\mathfrak{osp}(V)$ inside $V^{\otimes n}$.

The goal of the work [37] with Marin was to define a similar algebra for the Links-Gould polynomial. We first consider the corresponding centralizer algebra LG_n and prove the following statement, analogous to the well-known fact that the BMW_n algebra, defined as a (quantum) centralizer algebra, is a quotient of KB_n.

Theorem 2.1. *The natural morphism $KB_n \to LG_n$ is surjective.*

We do no expand here on the Lie theoretic and quantum background involved in the definition of the centralizer algebra LG_n and neither do we develop the proof of this theorem but rather refer here to [37] and for a more general setting where centralizer algebras can be seen as quotient of braid groups algebras [30].

As a consequence, LG_n is a natural candidate for being an analogue of the BMW_n algebra for the Links-Gould polynomial. As it is a centralizer algebra, we have a natural (combinatorial) description of its simple modules, but we do not have yet a satisfactory description of its elements.

Our main goal was to get a presentation of LG_n by generators and relations. We do no succeed completely, but we defined a finite dimensional quotient A_n of the cubic Hecke algebra and prove that it supports a unique Markov trace yielded by the Links-Gould polynomial that factorizes through LG_n.

2.1.2 Finite dimensional quotients of the cubic Hecke algebras

From the previous theorem we know that we are looking at finite dimensional quotients of the braid group algebra which are also a centralizer algebras for a certain family of 4-dimensional representations of the quantum Lie superalgebra $\mathcal{U}_q\mathfrak{so}(2|1)$. From this, one can compute the dimension of the algebra LG_n for small values of n and we have the following conjecture about its dimension in general:

Conjecture 2.1. *For all n,*

$$\dim LG_{n+1} = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}$$

We checked this formula for $n \leq 50$. It gives $\dim LG_1 = 1$, $\dim LG_2 = 3$, $\dim LG_3 = 20$, $\dim LG_4 = 175$, $\dim LG_5 = 1764$.

From the fact that we are looking at a finite dimensional quotient of the braid group algebra which satisfies a cubic relation one can use explicit descriptions of the cubic Hecke algebras $H_n = H_n(a, b, c)$ for $n \leq 5$. The cubic Hecke algebra $H_n = H_n(a, b, c)$ is the quotient of the group algebra KB_n of the braid group by the relations $(s_i - a)(s_i - b)(s_i - c) = 0$ or, equivalently, by the single relation $(s_1 - a)(s_1 - b)(s_1 - c) = 0$, since all s_i's are conjugated in B_n. In case a, b, c are three distinct roots of 1, H_n is the group algebra of the group $\Gamma_n = B_n/s_i^3 = B_n/s_1^3$, which is known to be finite if and only if $n \leq 5$, by a theorem of Coxeter (see [10]).

The algebras H_n are semi-simple and thus isomorphic over the algebraic closure $\overline{K}$ of K to a direct sum of matrix algebras. Moreover we can consider LG_n as a quotient of H_n, that is $LG_n = H_n/\mathfrak{I}_n$ for an ideal $\mathfrak{I}_n$ over H_n. Combining the explicit description of the algebras H_n for $n \leq 5$ (given for instance [7] see also [34] and [33]), work of Ishii [21] describing LG_3 by generators and relations and the computations of the dimensions of LG_n for small n, we obtained explicit matrix description of LG_n for $n \leq 5$.

In particular, in [21], Ishii introduced a relation $r_2 \in H_3$ satisfied by the Links-Gould polynomial and he proved that LG_3 is the quotient of H_3 by this relation r_2. From our study of LG_n and H_n for $n \leq 4$ we have the following :

Proposition 2.1. *The quotient of KB_4 by a generic cubic relation r_1 and Ishii's relation r_2 has dimension* $\dim KB_4/(r_1, r_2) = 264 > \dim LG_4 = 175$.

This proves that *Ishii's relations are not sufficient* to define LG_n. Using computer computations we were able to find a new relation $r_3 \in H_4$ such that

Proposition 2.2. $KB_4/(r_1, r_2, r_3) = H_4/(r_2, r_3) = LG_4$.

We obtained the relation r_3 by finding explicit bases of LG_3 and LG_4 in terms of braid words. In particular we have:

$$LG_4 = \left(\sum_{r \in \{-1,0,1\}} LG_3 s_3^r LG_3 \right) \oplus K s_3^{-1} s_2 s_3^{-1}$$

All these computations culminate in the following theorem :

Theorem 2.2. *For $n \geq 3$ we have*

1. $LG_n = LG_{n-1} s_{n-1}^{\pm 1} LG_{n-1} + \sum_{k+\ell=n} LG_k LG_l$
2. $LG_n = \sum_r LG_{n-1} s_{n-1}^r LG_{n-1} + LG_{n-3}(s_{n-1}^{-1} s_{n-2} s_{n-1}^{-1})$

We define $A_n = H_n/(r_2, r_3)$ for $n \geq 4$, $A_3 = H_3/(r_2)$, $A_2 = H_2$. By definition, $A_n \simeq LG_n$ for $n \leq 4$. The proof of the previous theorem relies on the explicit bases of $LG_3 = A_3$ and $LG_4 = A_4$ and the relation r_3, hence we have the same theorem where LG_n is replaced by A_n. In particular,

Theorem 2.3. *For all $n \geq 3$,*

$$A_{n+1} = A_n + A_n s_n A_n + A_n s_n^{-1} A_n + A_{n-2} s_n^{-1} s_{n-1} s_n^{-1}.$$

This implies immediately that A_n is finite dimensional for all n. We conjecture that A_n and LG_n have the same dimension and hence:

Conjecture 2.2. *For all n, $A_n \simeq LG_n$.*

2.1.3 Markov traces

The main result of this subsection is that the tower of algebras $(A_n)_{n\geq 1}$ can be endowed with a unique Markov trace Tr_n which computes the Links-Gould invariant. In addition we prove that the relations r_1, r_2 and r_3 are a complete set of relations for the Links-Gould invariant, i.e. one can recursively compute the Links-Gould invariant using this relations.

Given an integer $n \geq 1$, consider the natural embedding of B_n into B_{n+1}. Denote by ϕ_n its extension to an homomorphism from A_n to A_{n+1}.

Theorem 2.4. *For $z \in K$, there exists a family of traces $Tr_n : A_n \to K$, $n \geq 1$, such that*

- $Tr_{n+1}(\phi_n(\beta)) = zTr_n(\beta)$ *for all* $\beta \in A_n$.
- $Tr_n(\alpha\beta) = Tr_n(\beta\alpha)$ *for all* $\alpha, \beta \in A_n$.
- $Tr_{n+1}(\phi_n(\beta)s_n^{\pm 1}) = Tr_n(\beta)$ *for all* $\beta \in A_n$ *('Markov property').*
- $Tr_1 = 1$

if and only if $z = 0$. Moreover in this case, this family is unique. This trace factorizes through LG_n and is also the unique one on LG_n.

Remark 2.1. *Given an integer $n \geq 1$, for all $1 \leq k \leq n$ consider the natural embedding of $B_k \times B_{n-k}$ into B_n (see Figure (2.1)). Denote by ϕ_k its extension to an homomorphism from $A_k \otimes A_{n-k}$ to A_n. Define I_n the subvectorspace of A_n generated by the images of the ϕ_k ($1 \leq k \leq n$).*

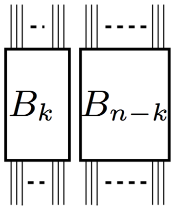


Figure 2.1: Injection of $B_k \times B_{n-k}$ into B_n.

The fact that z is equal to zero and an induction argument shows that the unique trace Tr_n on A_n vanishes on I_n. This implies that the Links-Gould invariant vanishes on split links (see [19] for a different proof).

Corollary 2.1. *The relations r_1, r_2 and r_3 are a complete set of skein relations for the Links-Gould invariant.*

Remark 2.2. *Notice first that the relations r_1, r_2 and r_3 are sufficient to compute the Links-Gould invariant. In addition, one can deduce relations expressing the elements $s_3^{\pm} s_2^{-1} s_1 s_2^{-1} s_3^{\pm}$, in the chosen basis of A_4 which could in pratice simplify a recursive computation. All these relations are of course consequences of r_1, r_2 and r_3*

Let us also mention that the Lawrence-Krammer representation factorize through LG_n.

Proposition 2.3. *For $n \geq 2$, the Lawrence-Krammer-Bigelow representation factorizes through LG_n and therefore, for $n \geq 2$, the morphism $B_n \to LG_n^{\times}$ is into.*

The proof of the proposition is obtained by giving a characterization the Lawrence-Krammer-Bigelow representation (see [37]).

In order to connect the present work, with the one presented in the next section, we stress here the first hypothesis satisfied by the Markov trace considered in Theorem 2.4. This miltiplicative property is not required from the definition of a Markov trace in general. It is nevertheless a property satisfied by all Markov traces constructed using representation theory of quantum groups. It is an easy check to see it follows from the quadratic relation (together with the 'Markov property') in the case of the Hecke algebra, but it is not a priori required for the Markov traces factoring through a cubic quotient of the braid group algebra. We do not investigate this question in the case of LG_n or A_n but first in the easier case of the BMW algebra which is also a finite dimensional quotient of the cubic Hecke algebra.

2.2 On the BMW algebras

In this section we present our joint work [36] with Marin aiming at classifying the Markov traces on the BMW algebra without assuming a multiplicative property. The first problem was to fix our definition of the BMW algebra and we introduced a presentation (see Definition 2.2) which was valid simultaneously for the symplectic and orthogonal version of the BMW algebra (seeing the BMW as a centralizer algebra). It turned out that for generic values of the parameters, it was still a version of the usual BMW algebra, but for some specializations it was one dimension bigger and this is the origin of the algebras presented in the next subsection. In this manuscript we present the results in the inverse order than the one in [36].

2.2.1 New finite dimensional quotients of the braid group algebra

We present in this subsection various finite dimensional quotients of the braid group algebra which are related to classical algebras such as Temperley-Lieb algebra, Hecke algebra and Birman-Murakami-Wenzl algebra. We explicit in the next subsection how they were constructed and explicit their relations with these classical algebras.

Theorem-Definition. *We define an algebra F_n over $A = \mathbb{Q}[a, x, x^{-1}]/(a^2 = 1)$ by generators $s_1, \dots, s_{n-1}$, $e_1, \dots, e_{n-1}, C$ and relations*

1. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ *if* $|i-j| > 1$
2. $(s_i - a)(s_i^2 - x s_i + 1) = 0$
3. $e_i = a\left(\frac{s_i^{-1}+s_i}{x} - 1\right)$
4. $s_i e_i = a e_i$
5. $e_i s_{i+1} e_i = e_i + C$
6. $e_i s_{i-1} e_i = e_i + C$
7. $e_i s_{i+1}^{-1} e_i = e_i + C$
8. $e_i s_{i-1}^{-1} e_i = e_i + C$
9. $s_i C = C s_i = aC$.

It's a free A-module of rank $1 + (2n-1)!!$.

Remark 2.3. *The quotient of F_n by the ideal generated by C is a specialization of a BMW-algebra.*

Remark 2.4. *Letting $\tilde{\delta} = 2 - ax$, we have $e_i^2 = \tilde{\delta} x^{-1} e_i$. Immediate consequences of these relations are $s_i^{-1} C = C s_i^{-1} = aC$, $e_i C = C e_i = x^{-1}\tilde{\delta} C$, $C^2 = (x^{-2}\tilde{\delta}^2 a - x^{-1}\tilde{\delta})C = x^{-1}\tilde{\delta}(a x^{-1}\tilde{\delta} - 1)C = 2x^{-2}\tilde{\delta}(a - x)C$. Note that, in the specializations $x = a$ and $x = 2a$, we have $C^2 = 0$. These are exactly the cases where F_n is a non trivial extension of the BMW-algebra.*

As in the usual case, one can consider a subalgebra corresponding to a Temperley-Lieb algebra and a quotient algebra corresponding to a Hecke algebra.

Theorem-Definition. *We define a unital algebra* $\widetilde{TL}_n$ *over* $A = \mathbb{Q}[a, x, x^{-1}]/(a^2 = 1)$ *by generators* $e_1, \ldots, e_{n-1}, C$ *and relations*

1. $e_i^2 = \tilde{\delta}x^{-1}e_i$
2. $e_iC = Ce_i = \tilde{\delta}x^{-1}C$
3. $C^2 = 2x^{-2}\tilde{\delta}(a - x)C$
4. $e_ie_j = e_je_i$ *if* $|j - i| \geq 2$
5. $e_ie_je_i = e_i + \frac{2a}{x}C$ *if* $|j - i| = 1$

It's a free A-module of rank $1 + \frac{1}{n+1}\binom{2n}{n}$.

Remark 2.5. *The quotient of* $\widetilde{TL}_n$ *by the ideal generated by C is a specialization of a Temperley-Lieb algebra.*

In order to obtain the Hecke algebra, one usually takes the quotient by the ideal generated by the e_i's, but by doing so we also have $C = 0$. Instead, we consider the quotient by the square of the ideal generated by the e_i's and C. In this quotient we have also $e_i = -C$ for all $i = 1, \ldots, n - 1$. Computing $e_iC = C^2$ in two different ways shows that C is still zero except if $x = 2a$. Set $A_0 = A/(x - 2a)$.

Theorem-Definition. *Consider* $F_n(0) = F_n \otimes_A A_0$ *and let* $A_0F_n^+$ *be the two sided ideal of* $F_n(0)$ *generated by* $e_1, \ldots, e_{n-1}, C$*. Define* $\overline{F}_n$ *to be the quotient of* $F_n(0)$ *by the ideal* $(A_0F_n^+)^2$*. It's a free* A_0*-module of rank* $1 + n!$.

We understands the structure of this algebra and it turns out that one can define a similar algebra for each Coxeter system (W, S), as we do now :

Theorem-Definition. *Let* (W, S) *be a Coxeter system, and k a field of characteristic* $\neq 2$*. The formulas*

$$\begin{cases} s.E_w &= E_{sw} & \text{if } \ell(sw) = \ell(w) + 1 \\ &= -2a^{\ell(w)}C + 2aE_w - E_{sw} & \text{otherwise} \\ s.C &= aC \end{cases}$$

for all $s \in S$*,* $w \in W$*, define a representation of the Artin-Tits group B associated to* (W, S) *on the free module over* $k[a]/(a^2-1)$ *spanned by C and the* E_w*,* $w \in W$*. When W is finite, the image of the group algebra of B inside this representation is a free module of rank* $1 + |W|$*. In all cases, this image projects onto the Iwahori-Hecke algebra of* (W, S) *defined by the relation* $(s-a)^2 = 0$ *for all* $s \in S$*, with kernel the linear span of* $\tilde{C} = -(s - a)^2$ *for an arbitrary choice of* $s \in S$*. When W admits a single conjugacy class of reflections, this algebra is the quotient of the group algebra of B by the relations* $(t - a)(s - a)^2 = (s - a)^2(t - a) = 0$ *for all* $s, t \in S$*.*

Remark 2.6. *The quotient of* $\overline{F}_n$ *by the ideal generated by C is the* (-1)*-Hecke algebra.*

In the next subsection we investigate various traces on these algebras.

2.2.2 Traces and Markov traces on F_n, $\overline{F}_n$ and $\widetilde{TL}_n$.

Since F_n has as a quotient a BMW algebra, we know already that there are at least two Markov traces, one t_n^H yielded by the HOMFLYPT polynomial and another t_n^K yielded by the Kauffman (or Dubrovnik) polynomial (see next subsection) Both traces t_n^H and t_n^K are by definition zero on C. In addition we have a trace $t_n^{\dagger\dagger}$ on F_n given by $t_n^{\dagger\dagger}(\beta) = a^n\psi_n(\beta)$, where $\psi_n : F_n \to A$ is an algebra morphism defined by $s_i \mapsto a$.

Theorem 2.5. *Given a Markov trace on* $F_n \otimes_A A[(x - a)^{-1}, (x - 2a)^{-1}]$ *, it is a linear combination of* t_n^H*,* t_n^K *and* $t_n^{\dagger\dagger}$*.*

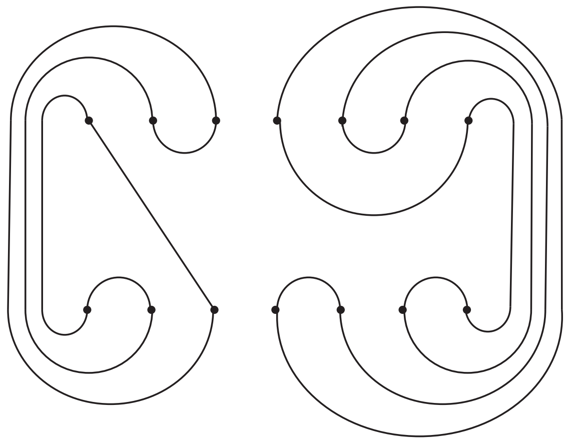


Figure 2.2: Closure of a Temperley-Lieb diagram with 2 components.

It can be seen that in this case $t_n^{\dagger\dagger}(C) \neq 0$.

It remains to study the cases $x = a$ and $x = 2a$. In the first case, the traces t_n^K and $t_n^{\dagger\dagger}$ coincide and in the second so do t_n^H and $t_n^{\dagger\dagger}$. We know then that in each case there could be at most three different Markov traces (see [36]. We would like to construct an additional Markov trace in each case. We do so in the case $x = 2a$ and conjecture the existence of a third Markov trace in the case $x = a$. Before pursuing, we consider the restriction of the additionnal trace on $\widetilde{TL}_n$ in each case.

Set $A_0 = A/(x-2a)$ and $A_1 = A/(x-a)$. Consider $\widetilde{TL}_n(0) = \widetilde{TL}_n \otimes_A A_0$, $\widetilde{TL}_n(1) = \widetilde{TL}_n \otimes_A A_1$.

Proposition 2.4. *There exist a family of traces $t_n : \widetilde{TL}_n(1) \to A_1$ satisfying $t_n(C) = -a^{n+1}$, and*

$$t_n(e_{i_1} \dots e_{i_k}) = a^{k+n} \left(\mathcal{N}(\bar{e}_{i_1} \dots \bar{e}_{i_k}) - k\right)$$

where $\mathcal{N}(\bar{e}_{i_1} \dots \bar{e}_{i_k})$ denotes the number of connected components of the diagrammatic closure of $\bar{e}_{i_1} \dots \bar{e}_{i_k} \in TL_n$ (see Figure 2.2)

Proposition 2.5. *Let $n \geq 3$ and $u_n, v_n \in A_0$. There exists a trace on $\widetilde{TL}_n(0)$ defined by $t_n(1) = v_n$, $t_n(C) = -u_n$, $t_n(e_i) = u_n$ for all $i \in [1, n-1]$ and*

$$t_n(e_{i_1} \dots e_{i_k}) = 0 \text{ if } k \geq 2.$$

Let us notice now that in the case $x = 2a$, we define in the previous subsection a finite dimensional quotient $\overline{F}_n$ of $F_n(0) = F_n \otimes_A A_0$. The next theorem settles the case of the Markov traces on $\overline{F}_n$.

Theorem 2.6. *There exists a unique family of traces $t_n : \overline{F}_n \to A_0$ satisfying $t_{n+1}(\beta s_n^{\pm 1}) = t_n(\beta)$ for all $\beta \in \overline{F}_{n-1}$ and $t_2(C) = 1$.*

The previous Markov trace has the following value on C: $t_n(C) = a^n$, for all n.

Corollary 2.2. *A Markov trace on $F_n(0)$ is a linear combination of $t_n^H = t_n^{\dagger\dagger}$, t_n^K and t_n.*

The Markov trace on $F_n(0)$ factoring through $\overline{F}_n$ coincides with the one constructed on $\widetilde{TL}_n(0)$ for the values $u_n = -a^n$, $v_1 = 0$ $v_n = (n-2)a^{n+1}$. Notice in particular that $t_1(1) = 0$ and $t_n(1) = (n-2)a^{n+1}$ for $n \geq 2$. Notice that we were able to construct the additional trace on $F_n(0)$ because there was a proof of the existence of the HOMFLYPT polynomial by Jones via the Ocneanu trace and this proof could be adapted in your setting. The situation for $F_n(1)$ is different, since there were only two proofs for the existence of the Kauffman polynomial, one using skein theory and diagrams by Kauffman and another using the representation theory of quantum groups. We needed a purely algebraic proof of the existence of the Kauffman polynomial in order to adapt it to our situation. This is why we only conjectured existence in [36]:

Conjecture 2.3. *There exist a Markov trace e on $F_n(1)$ such that $t_n(C) = -a^{n+1}$.*

In a joint work in progress with Poulain d'Andecy and Thiel [42] we constructed a very nice inductive basis of the BMW algebra which allows to give a positive answer to this conjecture, see the end of this section for further precisions.

2.2.3 Classification of the Markov traces on the BMW-algebras.

We explain now the relationship between the algebra F_n considered in the two previous sections and $BMW-algebra$. We give first two different definitions of a BMW-algebras. Set $R = \mathbb{Q}[a, b, c, (abc)^{-1}, (b+c)^{-1}]$ and denote $x = b + c$ and $y = bc$. Let RB_n be the braid group algebra on n strands over R. Define for $i = 1, \dots, n-1$.

$$e_i = \frac{a}{y}\left(\frac{ys_i^{-1} + s_i}{x} - 1\right).$$

Definition 2.1. *We define an algebra BMW_n over R by generators $s_1, \dots, s_{n-1}$, $e_1, \dots, e_{n-1}$ and relations*

1. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_j$
2. $(s_i - a)(s_i - b)(s_i - c) = 0$
3. $e_i = \frac{a}{y}\left(\frac{ys_i^{-1}+s_i}{x} - 1\right)$
4. $e_i s_{i+1} e_i = e_i$
5. $e_i s_{i-1} e_i = e_i$
6. $e_i s_{i+1}^{-1} e_i = e_i$
7. $e_i s_{i-1}^{-1} e_i = e_i$

Remark 2.7. *The algebra $BMW_n^+ = BMW_n \otimes_R R/(a = y)$ is the (up to change of coefficients) usual BMW algebra yielding the Kauffman polynomial. The algebra $BMW_n^- = BMW_n \otimes_R R/(a = -y)$ is the (up to change of coefficients) usual BMW algebra yielding the Dubrovnik polynomial.*

We consider now another quotient.

Definition 2.2. *We define an algebra $\widetilde{BMW}_n$ over R by generators $s_1, \dots, s_{n-1}$, $e_1, \dots, e_{n-1}$ and relations*

1. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i s_j = s_j s_i$ *for* $|i - j| > 1$
2. $(s_i - a)(s_i - b)(s_i - c) = 0$
3. $e_i = \frac{a}{y}\left(\frac{ys_i^{-1}+s_i}{x} - 1\right)$
4. $s_{i+1}^{-1} e_i s_{i+1}^{-1} = y^{-2} s_i e_{i+1} s_i$

Proposition 2.6. *The two algebras $BMW_n \otimes_R R/(a^2 - y^2)$ and $\widetilde{BMW}_n \otimes_R R/(a^2 - y^2)$ are naturally isomorphic after tensoring by $R[(bc - 1)^{-1}]$.*

The proof of this proposition amounts to prove that the relations on three stands generate the same ideal, if $a^2 = y^2$ and $a^2 - y$ invertible. First we consider the case of the algebra BMW_n.

Proposition 2.7. *There are at most two independent Markov traces on BMW_n.*

The proof of this proposition follows easily from the relation on three strands. In fact there are exactly two if $a^2 = y^2$ and one otherwise. In the first case they correspond to the Kauffman polynomial and HOMFLYPT polynomial, in the second only the HOMFLYPT remains. From the same type of reasoning there is only one Markov trace on $\widetilde{BMW}_n$ if $a^2 - y^2$ is invertible. It follows that the only remaining case to consider is the case of $\widetilde{BMW}_n$ when $a^2 = y^2 = y = 1$. This is where it connects to the first two subsections. Identify A with $R/(a^2 = y^2 = y = 1)$ and define the A-algebra $BMW_n^{\dagger\dagger} = \widetilde{BMW}_n \otimes_R /(a^2 = y^2 = y = 1)$

We have the following theorem:

Theorem 2.7. *The morphism of A-algebras $BMW_n^{\dagger\dagger} \to F_n$ induces an isomorphism after tensoring by $A[(x+2a)^{-1}]$.*

This isomorphism settles the question of the study of the Markov traces on BMW_n and on $\widetilde{BMW}_n$ up to the conjecture whose resolution we do in fact discuss in the next subsection and the case $x + 2a = 0$ on $\widetilde{BMW}_n$, for which one can see that there are infinitely many Markov traces detecting only the number of components of the closure of the braids, see [36].

2.3 Work in progress, open questions and perspectives

2.3.1 About the Links-Gould invariant

Even if we were able to find a relation on four strands for the Links-Gould invariant which was enough to give a complete set of skein relations for this invariant and to define a new finite quotient of the braid group algebra, this new relation is not enough manageable to further study this new quotient and understand deeper the Links-Gould invariant. The consequence is that in order to seriously attack our conjectures about the Links-Gould invariant and its defining algebra, we need an equivalent relation but more tractable.

Problem 2.1. *Find new skein relations for the Links-Gould invariant, in particular on four strands.*

One strategy to handle this problem is to look for instance for relations on four strands which will correspond to kind of lifts of relations on three strands in the sense that one obtains the latter by partially closing the former.

We mention also here the work of Ben-Michael Kohli who proved a conjecture of De Wit-Ishii-Links on a relation between the Links-Gould invariant and the Alexander polynomial and further conjectured that the Links-Gould invariant provides informations on the knot genus and the fiberness of knots. It naturally brings us to the following problem:

Problem 2.2. *Find a topological definition of the Links-Gould invariant.*

This question is in some sense the central question in quantum topology, but we believe that amongst all quantum invariants, the invariants coming from (super)-Hopf algebras are the most tractable ones from the topological point of view and the Links-Gould invariant the easiest one in this family of invariants.

2.3.2 About the BMW algebras

In the present version of my joint work with Marin [36], we only conjecture the existence of the additional non multiplicative Markov trace on $F_n(1)$, see Conjecture 2.3. We develop here our joint work in progress with Poulain d'Andecy and Thiel which as a by product settles this conjecture.

We construct inductively a new basis of the algebra BMW_n defined in 2.1 starting from $\mathbf{b}_1^{BMW} = \{1\}$ the basis of BMW_1. For $i = 1, \dots, n$, consider the following elements of BMW_{n+1}:

$$x_{n,i} = s_n{}^{-1} \cdots s_i{}^{-1} \text{ and } y_{i,n} = s_i \cdots s_n.$$

Then we have the following theorem :

Theorem 2.8. *The family $\mathbf{b}_{n+1}^{BMW} = \{\phi_n(b), \phi_n(b)x_{n,i}, y_{i,n}\phi_n(b) | b \in \mathbf{b}_n^{BMW}, i = 1, \dots, n\}$ forms a basis of BMW_{n+1}.*

Notice first that the cardinality of this basis is equal to the dimension of BMW_{n+1} and that the basis naturally generalizes the similar basis for the Hecke algebra (using only the $x_{n,i}$'s or the $y_{n,i}$'s). Moreover it does not descend to a basis of the Brauer algebra, when specializing the parameters, but its inductive construction allows to recover purely algebraically the two variable Kauffman polynomial.

The proof uses another basis of the BMW algebra constructed by Morton [39] and we prove by induction on n that each element of Morton's basis can be expressed in the inductive basis.

Corollary 2.3. *There exists exactly two Markov traces on the tower of algebras* $\left(BMW_n \otimes_R R/(a^2 - y^2)\right)_{n\geq 1}$.

Another corollary is a basis for the algebra F_n.

Corollary 2.4. *The family* $\{\mathbf{b}_n^{BMW} \cup C\}$ *forms a basis of* F_n

And hence a proof of conjecture 2.3:

Theorem 2.9. *Conjecture 2.3 is true.*

We finish this paragraph by mentioning that the Theorem 2.8 allows to classify the transverse Markov traces on the BMW algebras. Recall here that a transverse Markov trace, is only invariant under positive (de)-stabilization and is not required to be invariant under negative (de)-stabilization. The classification says that all transverse Markov traces are the one factoring through the Hecke algebra (which is a quotient of the BMW algebra) and the classical Markov trace yielding the two variable Kauffman polynomial. It is interesting to notice that the transverse Markov traces on the Hecke algebra are controlled by the HOMFLYPT polynomial and the self-linking number and this has a connection with the fact that some partial degree of the framing variable of the HOMFLYPT polynomial is related by an inequality (Morton-Franks-Williams inequality) to the self-linking number. This classification result on the Hecke algebra was surely known by the community. Nevertheless, a MFW-type inequality was not known for the two variable Kauffman polynomial which is now explained by the classification result on the BMW algebra.

2.3.3 About cubical quotients in general

Up to reparametrizations, there is only one quadratic quotient of the group algebra of the braid group, namely the Hecke algebra. In addition in this case there is only one Markov trace given by the HOMFLYPT polynomial a two variable link invariant. In the cubical case, as developed before, the situation is less clear and this brings us to the following question:

Problem 2.3. *Does there exist a finite dimensional quotient of the cubic Hecke algebra supporting a (multiplicative) three variable Markov trace?*

In the cases developed in this section, for the Links-Gould invariant and the two variable Kauffman polynomial, we know that the defining algebras do not support other multiplicative traces than the one we started with. We know also that the algebras BMW_n and LG_n are not directly related starting from $n = 4$ (for $n = 3$, BMW_3 is a quotient of LG_3). In addition we know by Ishii ([20]) that the Links-Gould invariant and the two variable Kauffman polynomial have a common specialization. A starting point, would be to understand a defining algebra for this common specialization and starting from there look for an algebra that will have both algebras as a quotient, at least on four strands. In addition one can also look for other quantum invariants satisfying a cubical relation. Let us finish by mentioning the recent work of Marin [35] which allows to see the Yokonuma-Hecke algebra as a cubical quotient, unfortunately the Markov traces on this algebra are now completely understood, see for instance [43].

Chapter 3

A little bit of categorification

3.1 Toward a categorification of the BMW algebras

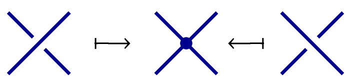
In this section we discuss the joint work with Vaz [53] whose aim was to explore connections between the BMW algebra (related to type B,C, D quantum groups by Schur-Weyl duality) and type A algebras (q-Schur algebras and $HOMFLYPT$ skein algebras of tangles). This connection was manifest in a formula by Jaeger relating the 2-variable Kauffman polynomial and the $HOMFLYPT$-polynomial. It was later further explored by Vaz as a manifestation of a branching rule. In the paper [53], we express this connection as a morphism of algebras and explore consequences from a categorification perspective. In this subsection, we concentrate for the A side on the skein algebra and refer to [53] for the development on the q-Schur algebra.

3.1.1 Skein algebras and BMW algebras.

Let $R = \mathbb{C}(a, q)$ be the field of rational functions in two variables and n a positive integer.
We introduce also some short-hand notation in order to simplify many of the expressions. For a formal parameter a and for $n, k \in \mathbb{Z}$ we denote $[a^n, k] := \frac{a^n q^{-k} - a^{-n} q^k}{q - q^{-1}}$. We allow a further simplification by writing $[a^n]$ instead of $[a^n, 0]$. Moreover, when dealing with 1-variable specializations we use $[m + k]$ for $[q^m, k] = \frac{q^{m+k} - q^{-m-k}}{q - q^{-1}}$, which is the usual *quantum integer*.

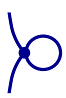
BMW algebras

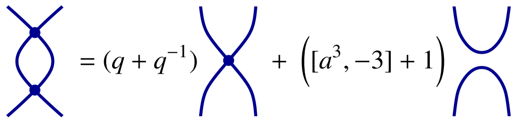
A (n, n) *4-graph* is a planar graph with $2n$ univalent vertices and such that the rest of the vertices are 4-valent. It can be embedded in a rectangle with n of the univalent vertices lying at the bottom segment and n lying at the top one. We think of a (n, n) 4-graph as the singularization of a (n, n) unoriented tangle diagram, which is the graph obtained by applying the transformation

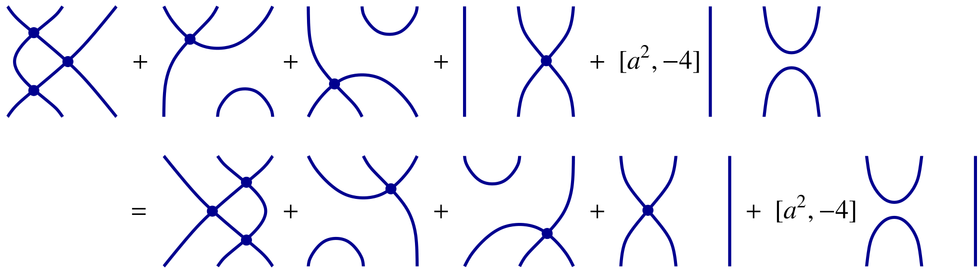


to all its crossings.

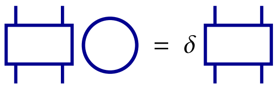
Definition 3.1. *Define* $\mathrm{BMW}_n(a, q)$ *as the free algebra over R generated by (n, n) 4-graphs up to planar isotopies fixing the univalent vertices modulo the following local relations:*

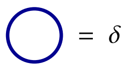
 $= [a](q^2 a^{-1} + q^{-2} a)$ 





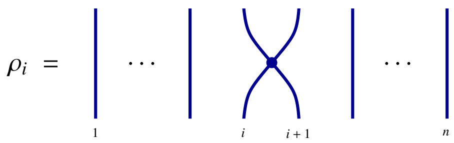
(3.1)



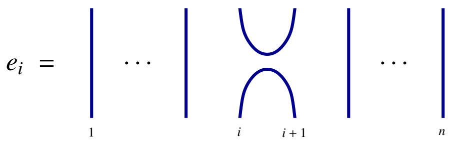


where $\delta = [a^2, -1] + 1$.

The product structure is given by stacking one graph over the other.
We remark that the special elements

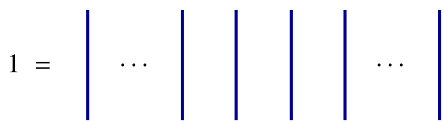


$(i = 1, \ldots, n-1)$ (3.2)



$(i = 1, \ldots, n-1)$ (3.3)

together with



(3.4)

generate $\mathrm{BMW}_n(a, q)$ and can be used to give a presentation of $\mathrm{BMW}_n(a, q)$ by generators and relations (see [39]) which makes explicit the isomorphism with the BMW algebras of the previous section.

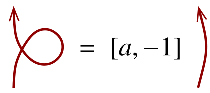
Skein algebras

We define an *oriented* (n, n) *4-graph* to be a (n, n) 4-graph together with an orientation looking near each 4-valent vertex as follows

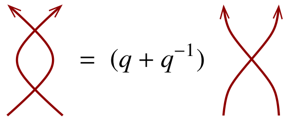
 .

In other words, an oriented (n, n) 4-graph is the singularization of a (n, n) oriented tangle diagram. All the oriented 4-valent graphs we consider are of this type.

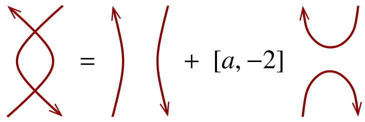
Definition 3.2. *Define the algebra* $\mathrm{Skein}_n(a,q)$ *as the free algebra over* R *generated by* (n,n) *oriented 4-valent graphs up to planar isotopies modded out by the following relations:*



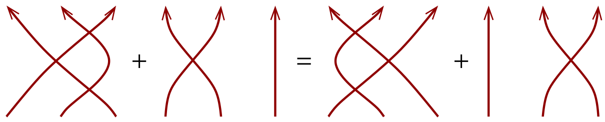
(3.5)



(3.6)

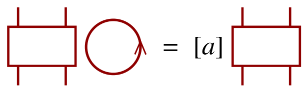


(3.7)



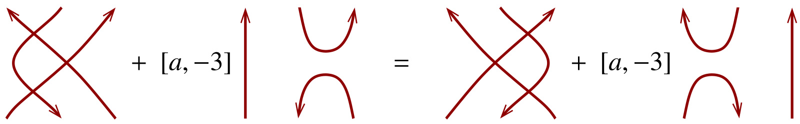
(3.8)

(3.9)



(3.10)

The product structure is given by stacking one graph over the other being zero if the orientations do not match. In addition,

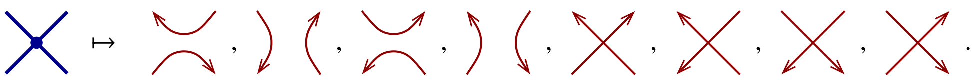


(3.11)

is a consequence of the previous relations.

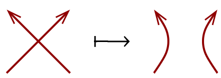
3.1.2 Connecting morphisms between Skein algebras and BMW algebras.

Given a (n,n) 4-valent graph Γ, we can resolve each of its vertex in eight different ways,

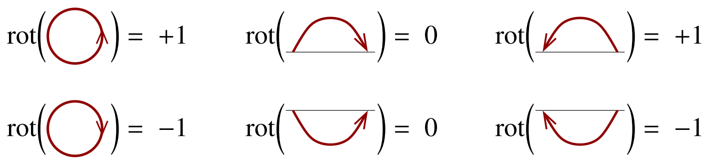


The graph obtained by choosing a resolution for each vertex is called a *complete resolution*. A complete resolution resulting in a coherently oriented (n,n) graph is called an *oriented complete resolution* and denoted $\vec{\Gamma}$. Denote by $\mathrm{res}(\Gamma)$ the set of all oriented complete resolutions of Γ. Notice that if there is no 4-valent vertex (i.e. Γ consists of an embedding of n arcs) there are 2^n resolutions consisting in choosing an orientation for each arc.

Given an oriented (n,n) 4-valent graph Γ we can apply the transformation



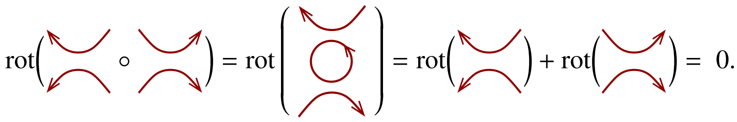
to smooth all 4-valent vertices of Γ and obtain a disjoint union of oriented circles and n oriented arcs embedded in the plane. Define the rotational number $\mathrm{rot}(\Gamma)$ of Γ to be the sum over all resulting circles and arcs of the contribution of each circle and each arc, where a circle contributes -1 if it is oriented clockwise and $+1$ otherwise, and arcs contribute ± 1 or zero according with the rules given below.



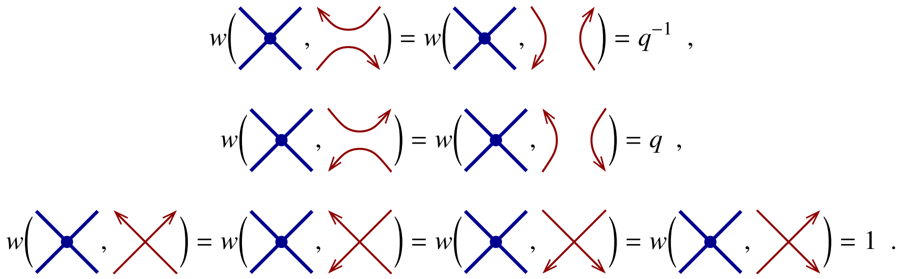
(3.12)

In addition the rotational number of a strand going up or down is zero.

The rotational number is additive with respect to the multiplicative structure of (n, n) 4-valent graphs given by concatenation. For example,



The last concept needed in this section is the *weight* w associated to each oriented complete resolution. It is computed as a product of local weights associated to each 4-valent vertex of Γ and an oriented resolution of it. The local weights are described below.



(3.13)

For any (n, n) 4-valent graph Γ, we define the *Jaeger homomorphism* as

$$\psi(\Gamma) = \sum_{\vec{\Gamma}\in\mathrm{res}(\Gamma)} (a^{-1}q)^{\mathrm{rot}(\vec{\Gamma})} w(\vec{\Gamma})\vec{\Gamma}. \tag{3.14}$$

Theorem 3.1. *The map ψ from $\mathrm{BMW}_n(a, q)$ to $\mathrm{Skein}_n(a, q)$ is a well-defined injective morphism of algebra.*

This morphism allows to see the BMW algebra as a subalgebra of a type A algebra. Based on this morphism, one can also obtain a morphism from the BMW algebra to direct sum of type A q-Schur algebras, we do not expand on this morphism and refer to [53].

3.1.3 A glimpse of categorification

We specialize the previous seting to $a = q^N$ and consider $\mathrm{Skein}_q(n, N) := \mathrm{Skein}_n(q^N, q)$. We denote also by $\mathrm{BMW}_q(n, N)$ the specialization $\mathrm{BMW}_n(q^N, q)$. This specialization is related to the representation theory of the quantum group $U_q(\mathfrak{so}_{2n})$.

Let us first recall the philosophy of categorification. The split Grothendieck group K_0 of an additive category C is the free abelian group generated by the isomorphism classes $[M]$ of objects M of C modulo the relation $[C] = [A] + [B]$ whenever $C \cong A \oplus B$. When C has a monoidal structure the Grothendieck group is a ring, with multiplication given by $[A \otimes B] = [A][B]$. Moreover, if C is a graded category then $K_0(C)$ has a structure of $\mathbb{Z}[q, q^{-1}]$-module, where $[M\{k\}] = q^k[M]$.

Let R be a commutative ring with 1, $\mathcal{A}$ and algebra over R and $\{a_i\}_{i\in I}$ a basis of $\mathcal{A}$. By a (weak) *categorification of* $(\mathcal{A}, \{a_i\}_{i\in I})$ we mean an additive monoidal category $\mathcal{A}$ together with an isomorphism

$$\gamma\colon R \otimes_{\mathbb{Z}} K_0(C) \to \mathcal{A} \tag{3.15}$$

sending the class of each indecomposable object of C to a basis element of $\mathcal{A}$ (see [25] for a detailed discussion).

Matrix factorizations and the $\mathrm{Skein}_q(n, N)$ categorification

In [27] Khovanov and Rozansky constructed a link homology theory categorifying the quantum $\mathfrak{sl}_N$-invariant P_N of links. The starting point is the diagrammatic MOY state-sum model [40] of P_N, whose underlying algebraic structure is exactly $\mathrm{Skein}_q(n, N)$ (this was the main motivation for the presentation given in Definition 3.2). The procedure consists expanding a link diagram D in an alternating sum in $\mathrm{Skein}_q(n, N)$, each term being evaluated to a polynomial in $\mathbb{Z}[q, q^{-1}]$ using the defining rules of $\mathrm{Skein}_q(n, N)$ from Definition 3.2.

The main ingredient of [27] is the use of matrix factorizations. Let R be a commutative ring and $W \in R$. A matrix factorization of W consists of a free $\mathbb{Z}/2\mathbb{Z}$-graded R-module M together with a map $D \in \mathrm{End}(M)$ of degree 1 satisfying $D^2 = W.\,\mathrm{Id}_M$.

In [27] Khovanov and Rozansky associated to each graph Γ in $\mathrm{Skein}_q(n, N)$ a certain graded matrix factorization $M(\Gamma)$ and showed that for each relation $\Gamma = \sum_i \Gamma_i$ in Definition 3.2 (with $a = q^N$) we have a direct sum decomposition $M(\Gamma) \cong \oplus_i M(\Gamma_i)$. To a link diagram they associated a complex of matrix factorizations and proved that the direct sum decompositions they obtain are sufficient to have topological invariance up to homotopy.

The reader now may ask why not use the bigraded matrix factorizations from [28] to obtain a categorification of the 2-variable BMW algebra. Unfortunately the matrix factorization from [28] associated to the left hand side of Equation (3.7) is not isomorphic to the direct sum of the matrix factorizations associated to the right hand side. This is the main reason why the HOMFLY-PT homologies that exist are defined only for braids and closures of braids and not for tangles.

Toward a categorification

We now explain abstractly our procedure. We use the symbol $\mathcal{Y}$ to refer to the categorification of $\mathrm{Skein}_q(n, N)$ and symbols $\{Y_j\}_{j\in J}$ to denote its indecomposable objects. This categorification has the *Krull-Schmidt property*, meaning that each object decomposes into direct sum of indecomposable objects which is unique up to permutation (see [44, Sec. 2.2]). This implies that the classes of the indecomposables in $K_0(\mathcal{Y})$ form a basis of $K_0(\mathcal{Y})$. In addition this basis is positive that is, all the multiplication coefficients in this basis are nonnegative since they count multiplicities in direct sum decompositions.

As explained before we expand every element x of $\mathrm{BMW}_q(n, N)$ as a linear combination of elements of another algebra, the latter admitting a categorification. We write it abstractly as

$$x = \sum_{j\in J} c_j y_j \tag{3.16}$$

where each y_j is a basis element of $\mathrm{Skein}_q(n, N)$ and $c_j \in \mathbb{N}[q, q^{-1}]$.

Homomorphism γ (3.15) sends $[Y_j]$ to y_j and therefore we think of the object Y_j as the lift to $\mathcal{Y}$ of the basis element y_j. This results in a well-defined object X of $\mathcal{Y}$ given by

$$X = \bigoplus_{j\in J} Y_j\{c_j\} \tag{3.17}$$

where we use the notation $Y\{q^{i_1} + \ldots + q^{i_k}\} = Y\{i_1\} \oplus \ldots \oplus Y\{i_k\}$.

We now define an additive monoidal category $\mathcal{X}$ from this data.

Definition 3.3. *Category $\mathcal{X}$ is the (monoidal) full subcategory of $\mathcal{Y}$ generated by products of the objects X given by Equations* (3.17) *which are images under Jaeger's homomorphism of the generators of* $\mathrm{BMW}_q(n, N)$ *from Equations* (3.2)-(3.4). *The morphisms of $\mathcal{X}$ are the obvious ones from $\mathcal{Y}$.*

Given a basis $(x_i)_{i\in I}$ of $\mathrm{BMW}_q(n,N)$ consider the element X_i constructed above for each x_i. Since the relations in $\mathrm{BMW}_q(n,N)$ lift to relations in $\mathcal{Y}$, it follows that the $\{[X_i]\}_{i\in I}$ generates the Grothendieck ring $K_0(\mathcal{X})$. Recall that $\mathrm{BMW}_q(n,N)$ is naturally equipped with a non-degenerate bilinear form given by the Kauffman polynomial. It follows that if there was non-trivial relations satisfied by the X_i's in $\mathcal{Y}$ it would contradict the non-degeneracy of this bilinear form. Hence we can deduce that the $[X_i]$'s are linearly independent in $K_0(\mathcal{X})$ and form a basis of $K_0(\mathcal{X})$. Using this remark and the results of [27] and [32] it is not hard to prove the following

Proposition 3.1. *We have an isomorphism* $K_0(\mathcal{X}) \cong \mathrm{BMW}_q(n,N)$.

Unfortunately the category $\mathcal{X}$ does not have the Krull-Schmidt property, which is a desirable property for the reason explained above. To get a categorification with the Krull-Schmidt property we need to add some objects to $\mathcal{X}$. This yields another category $\mathcal{X}'$ as follows.

Definition 3.4. *An object A of $\mathcal{Y}$ is an object of $\mathcal{X}'$ if there are objects B and C of $\mathcal{X}$ such that $A \oplus B \cong C$.*

The construction of category $\mathcal{X}'$ resembles the construction of the category of special bimodules in [48] (see also [38, Sec. 3.1]). Notice we still have $K_0(\mathcal{X}') \cong K_0(\mathcal{X})$. We were able to prove by hand that $\mathcal{X}'$ has indeed the Krull-Schmidt property in the cases up to $\mathrm{BMW}_q(3,N)$.

Conjecture 3.1. *The category $\mathcal{X}'$ has the Krull-Schmidt property.*

One could feel tempted to take the Karoubi envelope of $\mathcal{X}$ to guarantee Krull-Schmidt property. Recall the Karoubi envelope of a category $\mathcal{C}$ consists in adding more objects to $\mathcal{C}$ which are images of idempotents. In the Karoubi envelope every idempotent splits and consequently we have the Krull-Schmidt property [44]. It is easy to see that this procedure would add too many objects making the Grothendieck ring too large to be isomorphic to $\mathrm{BMW}_q(n,N)$.

We are suggesting a category having the Krull-Schmidt property which is not Karoubian. Such categories are known to exist. For example the category of super-vector spaces with odd dimension and even dimension both equal is not Karoubian but has the Krull-Schmidt property.

It would be interesting to relate the lift of the 4-vertex using matrix factorizations with the one Khovanov and Rozansky did in [26] using convolutions of matrix factorizations.

3.2 More categorification from topology

In this subsection we present our joint work with Gadbled and Thiel [14] on a categorification of a two parameters homological representation of the extended affine type A braid group. It is naturally an extension of the categorification of the Burau representation by Khovanov and Seidel [23]. The exposition differs from [14]; we do not include a complete definition of the homological representation we are categorifying and neither do we discuss (tri)-graded intersection numbers which was the heart of your proof of the faithfulness of the categorical action in [14]. The homological representation can be directly recovered from the categorical one, see Theorem 3.2 and Remark 3.4. We discuss in the paragraph after Theorem 3.3 and Corollary 3.1 another perspective on the faithfulness.

Let n be a fixed integer with $n \geq 3$.

3.2.1 Extended affine type A braid group

Braid groups by generators and relations

The extended affine type A braid group $\hat{\mathrm{B}}_{\hat{A}_{n-1}}$ is generated by

$$\sigma_1, \ldots, \sigma_n, \rho,$$

subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{for distant } i, j = 1, \ldots, n \tag{3.18}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad \text{for } i = 1, \ldots, n \tag{3.19}$$

$$\rho \sigma_i \rho^{-1} = \sigma_{i+1} \qquad \text{for } i = 1, \ldots, n \tag{3.20}$$

where the indices have to be understood modulo n, e.g. $\sigma_{n+1} = \sigma_1$ by definition. We say that i and j are distant (resp. adjacent) if $j \not\equiv i \pm 1 \mod n$ (resp. $j \equiv i \pm 1 \mod n$).

This group can be depicted by diagrams on the cylinder as shown in Figure 3.1 with the convention that a diagram drawn from bottom to top corresponds to a braid word read from right to left. The generator σ_i consists in a crossing between the strands labelled i and $i + 1$ modulo n while ρ consists in a cyclic permutation of the points 1 to n.

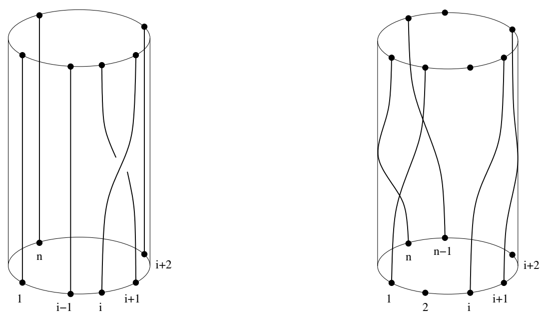


Figure 3.1: The affine braid generators σ_i and ρ

Remarks 1. *The group* $\hat{B}_{\hat{A}_{n-1}}$

- *possesses as subgroups the finite type A braid group* $B_{A_{n-1}}$ *generated by* $\sigma_1, \ldots, \sigma_{n-1}$*, but also the affine type A braid group* $B_{\hat{A}_{n-1}}$ *generated by* $\sigma_1, \ldots, \sigma_n$*. In fact,* $\hat{B}_{\hat{A}_{n-1}}$ *is simply isomorphic to the semi-direct product* $B_{\hat{A}_{n-1}} \rtimes \langle \rho \rangle$ *of the latter and of the infinite cyclic group generated by* ρ*, where the action of* ρ *on* $B_{\hat{A}_{n-1}}$ *given by conjugation permutes cyclically the generators* σ_i*;*

- *is isomorphic to the finite type B braid group* $B_{B_{n-1}}$ *generated by* $\sigma_1, \ldots, \sigma_{n-1}$ *and* τ *such that the* σ_i *are subject to the finite braid relations* (3.18) *for* $i = 1, \ldots, n-1$ *and* (3.19) *for* $i = 1, \ldots, n-2$ *and that the following relations are satisfied:*

$$\sigma_i \tau = \tau \sigma_i \qquad \text{for } i = 2, \ldots, n-1 \tag{3.21}$$

$$\tau \sigma_1 \tau \sigma_1 = \sigma_1 \tau \sigma_1 \tau. \tag{3.22}$$

This isomorphism identifies the generators σ_i *for* $i = 1, \ldots, n-1$ *while it sends* ρ *to the product* $\tau \sigma_1 \ldots \sigma_{n-1}$*;*

- *is a subgroup of the finite type A braid group* B_{A_n} *generated by the* σ_i *for* $i = 0, \ldots, n-1$ *subject to the finite braid relations* (3.18) *for* $i = 0, \ldots, n-1$ *and* (3.19) *for* $i = 0, \ldots, n-2$*. It consists exactly in the subgroup generated by the elements of* B_{A_n} *that leave the first strand (labelled by* 0*) fixed. One hence recovers the cylindrical pictorial description of* $\hat{B}_{\hat{A}_{n-1}}$ *by "inflating" this fixed strand that can be seen as the core of the cylinder, see e.g. Figure 3.2 depicting the image* $\sigma_0^2 \sigma_1 \ldots \sigma_{n-1}$ *of* ρ *in* B_{A_n}*. Note that the generator* σ_n *is sent to* $\sigma_0^2 \sigma_1 \ldots \sigma_{n-2} \sigma_{n-1} \sigma_{n-2}^{-1} \ldots \sigma_1^{-1} \sigma_0^{-2}$*, and that in the type B presentation of this group,* τ *is simply sent to* σ_0^2*.*

See [1], [52], [24] or [9] for more details about this subject.

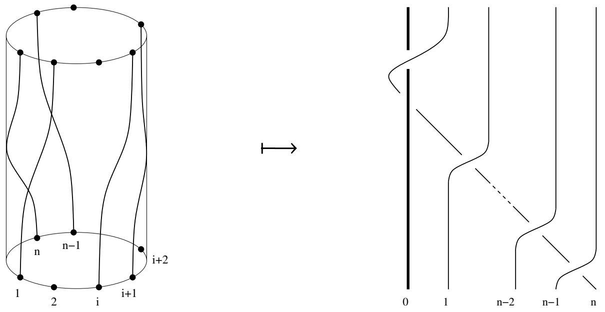


Figure 3.2: Image of ρ

Braid groups as mapping class groups

Let M be an orientable surface possibly with boundary. We will denote by $\mathrm{MCG}(M, n+1)$ the mapping class group of the surface M with $n+1$ marked points defined as the group of orientation-preserving homeomorphisms of M that fix the $n+1$ marked points setwise and the boundary pointwise up to isotopy. We will use the notation Δ for the set $\{0, \ldots, n\}$ of marked points and sometimes consider these marked points as punctures and view M as a $n+1$-punctured surface.

For a fixed set $S \subset \Delta$, we will also consider the subgroup $\mathrm{MCG}(M, n+1, S)$ of $\mathrm{MCG}(M, n+1)$ consisting of all mapping classes fixing pointwise the punctures of M belonging to the set S.

The finite type A braid group B_{A_n} is isomorphic to the mapping class group $\mathrm{MCG}(\mathbb{D}, n+1)$ of the $n+1$-punctured 2-disk $\mathbb{D}$ depicted in Figure 3.3.

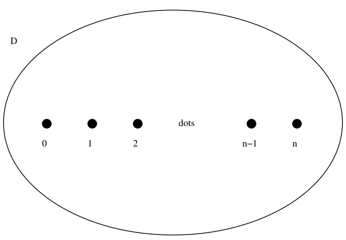


Figure 3.3: The $n+1$-punctured 2-disk $\mathbb{D}$

Each generator σ_i corresponds to the mapping class with support a small open disk enclosing the punctures i and $i+1$ and consisting in rotating this disk by π radians as described by Figure 3.4. We call this mapping class the half-twist along the arc b_i and denote it by t_{b_i}. It swaps the punctures i and $i+1$ and leaves all the others fixed pointwise.

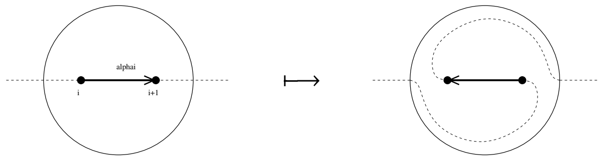


Figure 3.4: The half-twist along the arc b_i

Its subgroup $\mathrm{MCG}(\mathbb{D}, n+1, \{0\})$ is isomorphic to the finite type B braid group $\mathrm{B}_{B_{n-1}}$ where once again the half-twists t_{b_i} are identified to the generators σ_i for $i = 1, \ldots, n-1$, while τ corresponds to the full twist $t_{b_0}^2$. But $\mathrm{B}_{B_{n-1}}$ being isomorphic to $\hat{\mathrm{B}}_{\hat{A}_{n-1}}$, one might prefer to work with the extended affine type A presentation of this group. Then, to depict the generating mapping class corresponding to ρ, it is more convenient to draw the $n+1$-punctured 2-disk as in Figure 3.5. In this setting, ρ will simply correspond to the $1/n$-twist t_∂ with support an open disk enclosing all the punctures and consisting in rotating this disk by $2\pi/n$ radians. It sends the ith puncture to the $i+1$st mod n, for $i = 1, \ldots, n$ while leaving the 0th puncture fixed.

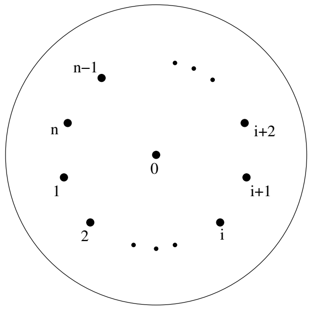


Figure 3.5: The affine configuration of the $n+1$-punctured 2-disk $\mathbb{D}$

In the sequel of this section, we will sometimes use indistinctly the same notation for a braid and a homeomorphism representing its mapping class.

In [29], Khovanov and Seidel construct a categorical representation of the finite type A braid group B_{A_n} in the homotopy category of the category of graded projective left modules over a certain quotient of the path algebra of a finite type A double quiver. This representation is faithful and it decategorifies on a linear one parameter representation equivalent to the Burau representation of B_{A_n}. Our aim is, following Khovanov and Seidel's ideas, to use an affine type A double quiver in order to obtain a categorical representation of the extended affine type A braid group $\hat{\mathrm{B}}_{\hat{A}_n}$. This latter representation is designed to decategorify on the linear 2-parameters homological representation similar to the classical Burau representation and hence requires to work with a module category endowed with a rich algebraic structure, namely a trigrading.

3.2.2 Action on a module category

This subsection is devoted to the definitions of those affine quiver algebra, trigraded module category and categorical representation.

The quiver algebra R_n

The notation used here for paths is taken from [29]. Recall that the path going from vertex i_1 to vertex i_k through the successive vertices $i_2, \ldots, i_{k-1}$ is denoted by $(i_1|i_2|\cdots|i_{k-1}|i_k)$. Start with the cyclic double quiver Γ_n pictured in Figure 3.6 and let R_n be the quotient of the path ring of the quiver Γ_n by the relations:

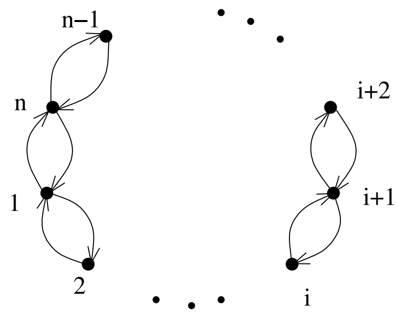


Figure 3.6: The affine type A double quiver Γ_n

$$(i|i+1|i) = (i|i-1|i) \qquad \text{for } i = 1, \dots n,$$
$$(i-1|i|i+1) = (i+1|i|i-1) = 0 \qquad \text{for } i = 1, \dots n,$$

where the integers are taken modulo n. This ring is trigraded and unital, with a family of mutually orthogonal idempotents (i) summing up to the unit element. It is generated as an algebra by the idempotents and the paths of length one. The three gradings on R_n are defined as follows:

- the first grading is defined by setting that the degree of $(i|i+1)$ is one while the degree of any other generator is zero (which is the opposite convention as the one chosen by Khovanov and Seidel);
- the second grading is simply the path length grading. Note that with the given relations, any path is at most of length 2 in R_n. This second grading will be considered as a $\mathbb{Z}/2\mathbb{Z}$ grading;
- the third grading is defined by setting that the degree of $(n|1)$ is 1, the degree of $(1|n)$ is -1 while the degree of any other generator in R_n is zero.

These three gradings are well-defined and one will denote by $\{-\}$ a shift in the first grading, by $(-)$ a shift in the second and by $\langle - \rangle$ a shift in the third. The convention being that the ith summand of a module shifted by k is the $(i-k)$th summand of the original module.

As an abelian group R_n is free of rank $4n$.

Remark 3.1. *If one forgets about the two last gradings on R_n, and just consider it as a singly graded algebra, it is in fact a particular case of the general construction of algebras associated to graphs by Huerfano and Khovanov, see [18]. Note that, in that paper, they are also considering actions of quantum groups and braid groups on certain module categories over these algebras.*

The category of finitely generated trigraded left modules R_n–mod has a Grothendieck ring $K(R_n - \text{mod})$ which is isomorphic to $\mathbb{Z}\left[t^{\pm 1}, s^{\pm 1}\right] \otimes \mathbb{Z}^n$. The $\mathbb{Z}\left[t^{\pm 1}, s^{\pm 1}\right]$-module structure comes from the self–equivalences $\{1\}$ and $\langle 1 \rangle$ of $R_n - \text{mod}$ consisting in shifting the first and third gradings by one. The problem with this category is that the isomorphism classes of the indecomposable left projective modules $P_i = R_n(i)$ do not form a basis of $K(R_n - \text{mod})$ as in the finite Khovanov-Seidel case because R_n has infinite global dimension.

Note that in the sequel, we will denote the right indecomposable projective modules ${}_iP = (i)R_n$ and the isomorphism class of a module M by $[M]$.

Therefore we will rather work over the category $R_n - \text{proj}$ of finitely generated trigraded projective left modules. This category, unlike $R_n - \text{mod}$, is not abelian, but only additive, though its split Grothendieck ring is also isomorphic to $\mathbb{Z}\left[t^{\pm 1}, s^{\pm 1}\right] \otimes \mathbb{Z}^n$, with basis $\mathbf{f} = \{[P_i], i = 1, \dots, n\}$. While, as before, the first grading (resp. the third) decategorifies onto the $\mathbb{Z}\left[t^{\pm 1}\right]$-module (resp. $\mathbb{Z}\left[s^{\pm 1}\right]$-module) structure, the second grading, which is a $\mathbb{Z}/2\mathbb{Z}$ grading, decategorifies as a sign (which will sometimes be denoted ϵ).

Let t_ρ be the automorphism of the ring R_n that sends any path $(i_1|i_2|\dots|i_k)$ to $(i_1+1|i_2+1|\dots|i_k+1)$. One can observe that this automorphism do not preserve the trigrading on R_n, but only the two first gradings. This implies that, if one constructs a bimodule R_n^ρ by simply twisting the right action on the regular bimodule R_n by t_ρ, i.e. $r \in R_n$ acts on R_n^ρ on the right by multiplication by $t_\rho(r)$, the resulting bimodule is not trigraded anymore. So, in order to define a trigraded twisted bimodule, one cannot only twist the action on the regular bimodule R_n but one has to construct a new bimodule in the more subtle way that we will describe now.

Let T_n^ρ be the trigraded R_n–bimodule generated by all elements of R_n set to be in the same first and second degree as in R_n, but with the third grading shuffled as follows:

- the degree of (1), $(2|1)$, $(1|n)$ and $(1|2|1)$ is -1
- the degree of any other generator is zero.

The left action of R_n on T_n^ρ is simply the multiplication while its right action is the multiplication twisted by t_ρ. Let us also consider the trigraded R_n–bimodule ${}^\rho T_n$ constructed similarly but with the third grading shuffling given by:

- the degree of (1), $(n|1)$, $(1|2)$ and $(1|2|1)$ is 1
- the degree of any other generator is zero.

Here R_n acts on the right on ${}^{\rho}T_n$ by multiplication and on the left by multiplication twisted by t_ρ.

Lemma 3.1. *T_n^ρ and ${}^{\rho}T_n$ are well-defined trigraded R_n–bimodules.*

Remark 3.2. *The chosen shufflings of the third grading appear to be natural when one observes that, as a trigraded left R_n–module, T_n^ρ is simply isomorphic to $P_1\langle -1\rangle \oplus P_2 \oplus \ldots \oplus P_n$ while, as a trigraded right R_n–module, ${}^{\rho}T_n$ is isomorphic to ${}_1P\langle 1\rangle \oplus {}_2P \oplus \ldots \oplus {}_nP$.*

Categorical representation

We consider C_n to be the homotopy category of bounded cochain complexes of R_n–proj. Its Grothendieck ring is also isomorphic to $\mathbb{Z}\left[t^{\pm 1}, s^{\pm 1}\right] \otimes \mathbb{Z}^n$, with basis the isomorphism classes of indecomposable projectives, see [45].

For all $i = 1, \ldots, n$, the two complexes F_i and F_i' of R_n–bimodules are defined as in [29]:

$$F_i : 0 \to P_i \otimes_{\mathbb{Z}} {}_iP \xrightarrow{d_i} R_n \to 0$$
$$F_i' : 0 \to R_n \xrightarrow{d_i'} P_i \otimes_{\mathbb{Z}} {}_iP\{-1\} \to 0$$

with R_n sitting in cohomological degree zero and where the respective differentials of these length one complexes are:

$$d_i((i)\otimes(i)) = (i)$$
$$d_i'(1) = (i-1|i)\otimes(i|i-1) + (i+1|i)\otimes(i|i+1) + (i)\otimes(i|i-1|i) + (i|i-1|i)\otimes(i)$$

where the integers again have to be understood modulo n.

Consider also the two following complexes of R_n–bimodules of length zero concentrated in cohomological degree zero:

$$F_\rho : 0 \to T_n^\rho \to 0$$
$$F_\rho' : 0 \to {}^{\rho}T_n \to 0$$

Remark 3.3. *Consider the functors*

$$\mathcal{F}_i = F_i \otimes_{R_n} -, \quad \mathcal{F}_i' = F_i' \otimes_{R_n} -, \quad \mathcal{F}_\rho = F_\rho \otimes_{R_n} - \text{ and } \mathcal{F}_\rho' = F_\rho' \otimes_{R_n} -.$$

Since the bimodules $P_i \otimes_{\mathbb{Z}} {}_iP$, T_n^ρ and ${}^{\rho}T_n$ are projective as left modules, the former functors are well-defined endofunctors of the category C_n. Plus these bimodules being also projective as right modules, these functors are actually exact. Hence they induce linear maps on the Grothendieck ring $K(C_n)$.

Theorem 3.2.

(i) There is a weak action of the braid group $\hat{B}_{\hat{A}_{n-1}}$ on C_n given on the generators σ_i by the functors $\mathcal{F}_i$, on their inverses σ_i^{-1} by the functors $\mathcal{F}_i'$, on the generator ρ by the functor $\mathcal{F}_\rho$, on its inverse ρ^{-1} by the functor $\mathcal{F}_\rho'$ and on any braid word σ by the functor $\mathcal{F}_\sigma$ consisting in tensoring on the left by the tensor product of the complexes associated to the generators appearing in the braid word.

(ii) This action induces a linear representation ρ_{AKS} of $\hat{B}_{\hat{A}_{n-1}}$ on the Grothendieck ring $K(C_n) \cong \mathbb{Z}\left[t^{\pm 1}, s^{\pm 1}\right]^n$ which is given in the basis $\mathbf{f} = \{[P_i], i = 1, \dots, n\}$ of $K(C_n)$ by:

$$\rho_{AKS}(\sigma_i) = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -t & t & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} \qquad \text{for } i = 2, \dots n-1,$$

$$\rho_{AKS}(\sigma_1) = \begin{pmatrix} -t & t & 0 & s^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{n-3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_{AKS}(\sigma_n) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ ts & 0 & 1 & -t \end{pmatrix}$$

$$\rho_{AKS}(\rho) = \begin{pmatrix} 0 & s^{-1} \\ I_{n-1} & 0 \end{pmatrix}$$

$$\rho_{AKS}(\rho^{-1}) = \begin{pmatrix} 0 & I_{n-1} \\ s & 0 \end{pmatrix}$$

Remark 3.4. *The linear representation obtained by decategorification ρ_{AKS} is equal to the homological action of the extended affine braid group on the first homology of a $\mathbb{Z}^2$-cover of the punctured disk.*

3.2.3 faithfulness of the categorical action

Trigraded curves and normal forms

Consider the real projectivization $P = PT(\mathbb{D} \backslash \Delta)$ and a covering of it with deck transformation group $\mathbb{Z}^3$. Consider an oriented embedding of $\mathbb{D}$ as an open subset of $\mathbb{R}^2$ so that its tangent bundle $T\mathbb{D}$ has a canonical oriented trivialization. As a consequence the projectivization of $T\mathbb{D}$ in restriction over $\mathbb{D} \backslash \Delta$ identifies with

$$PT(\mathbb{D} \backslash \Delta) = \mathbb{R}P^1 \times (\mathbb{D} \backslash \Delta).$$

For any puncture i in Δ, we will denote by $\lambda_i : S^1 \to \mathbb{D} \backslash \Delta$ a choice of a small loop winding positively around the puncture i. As the classes $[point \times \lambda_i]$ together with the class of a fibre $[\mathbb{R}P^1 \times point]$ form a basis of $H_1(P; \mathbb{Z})$, we define a class $C \in H^1(P; \mathbb{Z}^3)$ by specifying its images on these elements, namely:

$$\begin{aligned} C([point \times \lambda_i]) &= (-2, 1, 0) \qquad \text{for } i = 1, \dots, n \\ C([point \times \lambda_0]) &= (-2, 0, 1) \\ C([\mathbb{R}P^1 \times point]) &= (1, 0, 0). \end{aligned}$$

We will denote by $\check{P}$ the covering classified by C and by χ the $\mathbb{Z}^3$-action on it.

Let f be an element of $\mathrm{Diff}(\mathbb{D}, \Delta, \{0\})$, its differential Df is a diffeomorphism of the tangent bundle of $\mathbb{D} \backslash \Delta$ which is linear in the fibres of $T(\mathbb{D} \backslash \Delta)$ and thus induces a diffeomorphism PDf of P. As such a map f preserves winding numbers and as Df sends a fibre to another fibre, the map PDf preserves the class C and can be lifted to an equivariant diffeomorphism of $\check{P}$. We will denote by $\check{f}$ the unique lift of PDf which acts trivially on the fibre of $\check{P}$ over any point of $P_{|\partial \mathbb{D}}$.

Note that any curve c has a canonical section $s_c : c \backslash \Delta \to P$ by taking the class in each fiber of its tangent line: $s_c(z) = [T_z c]$. One defines a trigrading of c to be a lift $\check{c}$ of s_c to $\check{P}$ and a trigraded curve to be a pair $(c, \check{c})$ of a curve and a trigrading of that curve. The $\mathbb{Z}^3$-action on $\check{P}$ induces a $\mathbb{Z}^3$-action on the set of trigraded curves and the lifts of diffeomorphisms induce a $\mathrm{Diff}(\mathbb{D}, \Delta, \{0\})$-action on this set that commutes with the $\mathbb{Z}^3$-action. One can also lift the isotopy relation so that these actions induce actions of these same group on the set of isotopy classes of trigraded curves.

The arguments of [29] adapt to our trigraded case so that we have the following properties:

Lemma 3.2.

(i) A curve c admits a trigrading if and only if it is not a simple closed curve.

(ii) The $\mathbb{Z}^3$-action on the set of isotopy classes of trigraded curves is free: a trigraded curve $\check{c}$ is never isotopic to $\chi(r_1, r_2, r_3)\check{c}$ for any $(r_1, r_2, r_3) \neq 0$.

(iii) Let c be a curve which joins two points of Δ, none of them being the puncture 0, let $t_c \in \mathrm{Diff}(\mathbb{D}, \Delta, \{0\})$ be the half twist along it and $\check{t}_c$ its preferred lift to $\check{P}$. Then $\check{t}_c(\check{c}) = \chi(-1, 1, 0)\check{c}$ for any trigrading $\check{c}$ of c.

Consider basic curves $b_1, \ldots, b_n$ and curves $d_1, \ldots d_n$ as in Figure 3.7 which divide the disc $\mathbb{D}$ into regions $D_1, \ldots, D_n$.

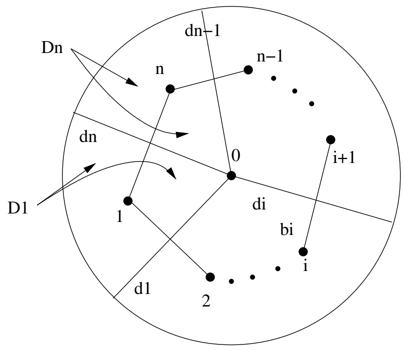


Figure 3.7: The arcs b_i and d_i and the sectors D_i.

A curve c is called admissible if there exist a mapping class σ and $i \in \{1, \ldots, n\}$ such that $c = \sigma(b_i)$. So the endpoints of any admissible curve are in $\{1, \ldots, n\}$. Conversely any curve whose endpoints lie in $\{1, \ldots, n\}$ is admissible.

We will say that an admissible curve c is in normal form if it has minimal intersection with all the d_i's. One can always achieve normal form by isotopy. The study of curves in this section makes sense because of the following uniqueness result:

Lemma 3.3. *Let c_0 and c_1 be two isotopic curves, both of which are in normal form. Then there is an isotopy relative to $d_1 \cup d_2 \cup \ldots \cup d_n$ which carries c_0 to c_1.*

Let c be a curve in normal form. Then each connected component of $c \cap D_k$ belongs to one of the six following types depicted in Figure 3.8.

Conversely, an admissible curve c which intersects all the d_k transversally and such that each connected component of $c \cap D_k$ belongs to one of the types listed in the Figure 3.8 is already in normal form.

For the rest of this section, c is an admissible curve in normal form. We will call crossing and denote $cr(c) = c \cap (d_1 \cup d_2 \cup \ldots \cup d_n)$ the intersections of this curve with the barriers of the sectors and the intersections with d_k will be called k-crossings of c. The connected components of $c \cap D_k, 1 \leq k \leq n$ are called segments of c, and a segment is said essential if its endpoints are both crossings (and not punctures). So, the essential segments are those of type 1, 1', 2, 2', and the basic curves have no essential segments.

The curve c can be reconstructed up to isotopy by listing its crossings and the types of essential segments bounded by consecutive crossings as one travels along c from one end to the other, and Lemma 3.3 shows that conversely this combinatorial data is an invariant of the isotopy class of c.

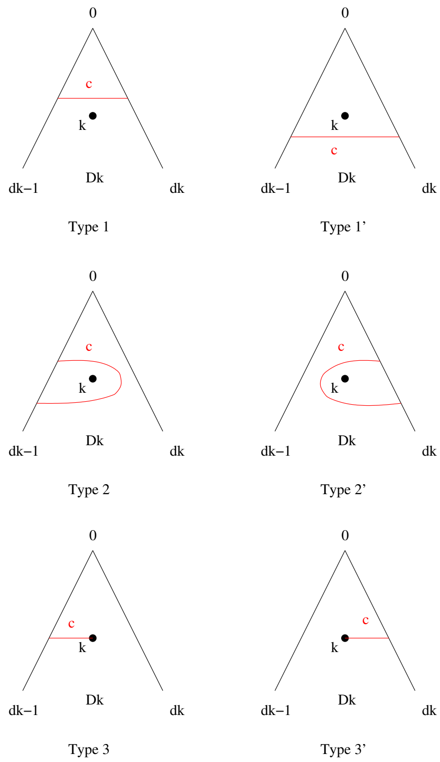


Figure 3.8: The six possible types.

Let us now study the action of half-twists on normal forms. Remember that we denoted t_{b_k} the half-twist along b_k. Even when the curve c is in normal form, the curve $t_{b_k}(c)$ is not necessarly in normal form too. This image $t_{b_k}(c)$ has minimal intersection with the d_i for $i \neq k$ but one might need to simplify its intersections with d_k to get $t_{b_k}(c)$ into normal form. The same argument as in [29] leads to the analogous statement:

Proposition 3.2.

(i) The normal form of $t_{b_k}(c)$ coincides with c outside of $D_k \cup D_{k+1}$. The curve $t_{b_k}(c)$ can be brought into normal form by an isotopy inside $D_k \cup D_{k+1}$.

(ii) Assume $t_{b_k}(c)$ is in normal form. There is a natural bijection between the i-crossings of c and the i-crossings of $t_{b_k}(c)$ for $i \neq k$. There is a natural bijection between connected components of intersections of c and $t_{b_k}(c)$ inside $D_k \cup D_{k+1}$.

Admissible curves and complexes of projective modules

Given a trigrading $\check{c}$ of an admissible curve in normal form, we associate an object $L(\check{c})$ in the category C'_n of bounded complexes of projective bigraded modules over the quiver algebra R_n. We define $L(\check{c})$ first as a trigraded R_n-module as follows:

$$L(\check{c}) = \bigoplus_{z \in cr(\check{c})} P(z),$$

where $P(z) = P_{k(z)}[-\mu_1(z) - n\mu_3(z)]\{\mu_2(z) - n\mu_3(z)\}\langle -\mu_3(z)\rangle$ with $[-]$ being a shift in the cohomological grading. We endow the previous trigraded module with a differential given by:

- If z_0 and z_1 are two boundary crossings of an essential segment then it follows that they differ in their μ_1 grading by 1. Suppose for instance that $\mu_1(z_1) = \mu_1(z_0) + 1$. There are two possibilities:

- If z_0 and z_1 are both k-crossings then $\partial : P(z_0) \to P(z_1)$ is the right multiplication by the element $(k|k+1|k)$.
- If z_0 (resp. z_1) is a k_0-crossing (resp. a k_1-crossing) and we have $|k_0 - k_1| = 1$, then $\partial : P(z_0) \to P(z_1)$ is the right multiplication by the element $(k_0|k_1)$.

- If z_0 and z_1 are not connected by an essential segment then there is no contribution of the differential between $P(z_0)$ and $P(z_1)$.

It can be directly checked that the previous map ∂ satisfies $\partial^2 = 0$ (this follows from the relations in the quiver algebra R_n) and in addition ∂ is of trigraded degree $(1, 0, 0)$, the first one corresponding to the cohomological degree and the other two to the internal degrees of C'_n. Hence we have the following lemma.

Lemma 3.4. *For all admissible curves $\check{c}$ in normal form, $(L(\check{c}), \partial)$ is a trigraded differential module.*

There is a free $\mathbb{Z}^3$-action on trigraded curves and also on the category C'_n (by shifts). The next lemma relates these two actions and can be directly checked from the construction of the trigraded differential module $L(\check{c})$.

Lemma 3.5. *For any triple (r_1, r_2, r_3) of integers and any admissible trigraded curve $\check{c}$ we have:*

$$L(\chi(r_1, r_2, r_3)\check{c}) \cong L(\check{c})[-r_1 - nr_3]\{r_2 - nr_3\}\langle -r_3 \rangle .$$

The aim of the next theorem is to relate the action by endofunctor of the extended affine type A braid group on $L(\check{c})$ and the complex associated to the image of the curve $\check{c}$ under the mapping class group action.

Theorem 3.3. *For any admissible trigraded curve $\check{c}$, we have the following isomorphisms in C'_n:*

$$\mathcal{F}_i(L(\check{c})) \cong L(\check{t}_{b_i}(\check{c})) \text{ for all } 1 \leq i \leq n.$$

and

$$\mathcal{F}_\rho(L(\check{c})) \cong L(\check{t}_\partial(\check{c})).$$

Corollary 3.1. *The categorical action is faithful*

The original proof of Khovanov-Seidel relies on the fact that the graded dimension of the space of morphisms of their category encodes graded geometric intersection numbers between curves. The faithfulness of the categorical action then follows from the fact that it by definition preserves these spaces of morphisms and and as a by product the (graded) geometric intersection numbers, and a mapping class that preserves the geometric intersection numbers between admissible curves, is easily proved to be the identity. The previous theorem proves that the subcategory C additively generated by the complexes associated to admissible trigraded curves is stable under the braid group action. It is not to difficult to see that for each trigraded admissible curve $\check{c}$ the complex $L(\check{c})$ is indecomposable. Moreover there is a one-to-one correspondance between indecomposable objects in C and trigraded admissible curves $\check{c}$. As a consequence, acting as the identity on the subcategory C implies acting as the identity on trigraded admissible curves, and the only braid which acts as the identity on (trigraded)-admissible curves is the identity and the faithfulness follows. In [14], we prove the faithfulness result using the graded dimension of spaces of morphism as Khovanov and Seidel did, but it is interesting to see Theorem 3.3 as the key step in the proof of faithfulness and we discuss this point of view further in the next section 3.3.2.

3.3 Work in progress, open questions and perspectives

3.3.1 Symplectic side of the picture

The original work of Khovanov-Seidel [23] was two-fold. In a first part they considered a zig-zag algebra, introduced graded intersection numbers and proved the faithfulness of their braid group categorical action. This is the part we generalized in our joint work with Gadbled and Thiel [14]. In the second part of their paper they considered the symplectic side of the picture, by looking to the Fukaya category of the A_n singularity. In this setting, the categorical braid action comes from symplectic Dehn twists along Lagrangian spheres, and they proved also faithfulness of this categorical action. The exact relationship between the two pictures was not completely established at that time but was confirmed in the work of Seidel and Thomas [47]. In a work in progress with Gadbled and Thiel we investigate the symplectic side of the picture. If we ignore one of the grading we introduced, and restrict for instance to the affine braid group of type A, the quadratic dual of our quiver algebra was used by Ishii-Ueda-Uehara [22] to establish the faithfulness of a categorical action using an Homological Mirror Symmetry statement and the strategy of proof of Khovanov-Seidel in the symplectic setting. It is not clear how to adapt their proof of their Homological Mirror Symmetry statement including the additional grading and we instead want to prove a formality statement which will allow us to prove that the algebraic category we are working with is equivalent to a certain Fukaya category. We already know how to endow the Lagrangians with a bigrading. The question of search for a mirror in this richer case (bigraded in the symplectic side) seems to be an interesting question on its own and we plan to investigate it later.

3.3.2 Categorification of the Burau representation and Lawrence-Krammer-Bigelow representations

The fact that Khovanov-Seidel categorical action was faithful unlike the Burau representation it categorifies was source of various natural questions. Recall also here that the categorical mapping class group action provided by the bordered Heegaard-Floer is also known to be faithful [31] with a very similar proof to the one of Khovanov-Seidel, we generalized in the previous section.

Problem 3.1. *Can algebraic properties of G be proved using the fact it has a faithful categorical action on a module category?*

This is a very interesting question but in the case of the braid group since there are so few faithfulness results it would be interesting to see how they interact. As developed in the previous section our point of view on the faithfulness of the categorical action is that it encodes in fact essentially the faithful action on the infinite dimensional vector space formally generated by isotopy classes of embedded arcs connecting two punctures. The process of decategorification transforms this faithful infinite dimensional linear representation into a finite dimensional unfaithful linear representation, the Burau representation. This brings us to the following question

Problem 3.2. *For the categorical braid group action of Khovanov and Seidel, find a finer decategorification process which retains more topological informations and remains faithfull.*

In this braid group case, the linearity was proved using the Lawrence-Krammer-Bigelow representation. Hence a more precise question would be if one can find a kind of decategorification which returns the Lawrence-Krammer representation. Preliminary computations with Queffelec and Thiel show that one can extract the Lawrence-Krammer-Bigelow representation from the categorical action of Khovanov-Seidel. For the moment the process is purely combinatorial and we seek for a more formal and algebraic one. These computations already indicate that the cohomological grading used by Khovanov and Seidel is related to the second variable in the Lawrence-Krammer-Bigelow representation.

3.3.3 Unitarity and categorification

Most of the known categorifications rely on the existence of a (preserved) sesquilinear form and the categorification of Khovanov-Seidel of the Burau representation is one of them. The Burau representation was proved to be unitarity by Squier [49] who explicitely gives the sesquilinear form and the matrix could a posteriori be obtained from the matrix whose entries are the graded dimensions of the space of morphisms between the projective indecomposables in Khovanov-Seidel construction. Moreover starting from this matrix one could have guessed which algebra to use in this very particular situation. It turns out that the

Lawrence-Krammer-Bigelow representation is also unitary and Budney [8] explicitely gave the sesquilinear form in this case. Using it one can attack the following question:

Problem 3.3. *Categorify the Lawrence-Krammer-Bigelow representation.*

Since the Lawrence-Krammer-Bigelow representation factorizes through the BMW algebra, a solution to the previous question would also shed some light on the categorification of this algebra.

Appendix:

We do not present the origin of the formula below, but it is directly connected to the joint work with Droz [11]. In this work we explicitely connected the Kauffman state model of the Jones polynomial and another model related to grid diagrams and Bigelow homological description of the Jones polynomial.

Given an oriented link diagram D, choose a base point p on D. An enhanced Kauffman state of D is a choice of a resolution for each crossing of D (see Figure 3.9), together with a choice of orientation on every resulting circle, see Figure 3.10 for an example.

A reduced enhanced Kauffman state of D is a choice of one resolution for each crossing of D, together with a choice of orientation on every resulting circle, such that the orientation of the circle passing through p agrees near p with the orientation of D. Denote by $\mathcal{K}^{\text{red}}$ the set of reduced enhanced Kauffman states.

Given an oriented link diagram D, we resolve all the crossings of D as in Figure 3.11, we obtain a disjoint union of oriented circles embedded in $\mathbb{R}^2$. We call these circles the *Seifert circles* of D. The *rotational number* of D, denoted by $\text{rot}(D)$, is the sum of the contributions of the circles. The contribution of a Seifert circle is $+1$ if it is oriented counterclockwise and -1 otherwise. Denote by $\text{rot}(p)$ the rotation number of the Seifert circle passing through p. A basepoint p is said *good* if can be connected to infinity without crossing D.

Given a crossing c of an oriented link diagram D, we define $w(c)$ as in Figure 3.12. We define the *writhe* $w(D)$ of D,

$$w(D) = \sum_{c \text{ crossings of } D} w(c).$$

We denote by n_+ the number of positive crossings and by n_- the number of negative crossings of D. We have $w(D) = n_+ - n_-$.

We define a grading $Alex : \mathcal{K}^{\text{red}} \to \mathbb{Z}$ by the local weights described in Figure 3.13. For each state $s \in \mathcal{K}^{\text{red}}$, $Alex(s)$ is the sum over the crossings of D of the local weights.

Moreover, for any $s \in \mathcal{K}^{\text{red}}$, define $\text{rot}(s)$ to be the sum of the rotational numbers of the oriented circles constituing s and $i(s)$ the number of resolutions in the underlying Kauffman state of the type depicted on the lefthandside of Figure 3.9 (the so-called 1 resolution). Set

$$Mas(s) = \frac{i(s)}{2} - \frac{\text{rot}(s)}{2} + Alex(s), \text{ for all } s \in \mathcal{K}^{\text{red}}.$$

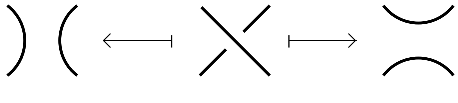


Figure 3.9: Kauffman resolutions

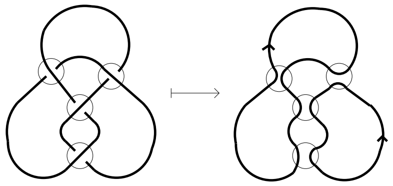


Figure 3.10: Example of an enhanced Kauffman state

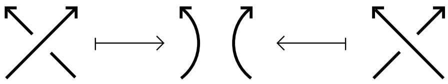
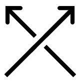


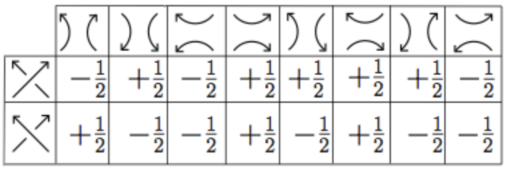
Figure 3.11: Oriented resolution

Negative crossing c, $w(c) = -1$



Positive crossing c, $w(c) = +1$

Figure 3.12: Crossings



	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Figure 3.13: Local weights for the grading *Alex*

Define

$$Q(t) = \sum_{s \in \mathcal{K}^{\text{red}}} (-1)^{Mas(s)} t^{Alex(s)}.$$

Theorem 3.4. *Given an oriented link diagram D and a good basepoint p, the polynomial*

$$\nabla(D)(t) = (-1)^{-\frac{n_+}{2} - \frac{\text{rot}(D)}{2} + \text{rot}(p)} t^{-\frac{\text{rot}(D)}{2} + \frac{\text{rot}(p)}{2}} Q(t),$$

is the Alexander-Conway polynomial.

The proof is completely classical. We check that

$$W(t) = (-1)^{-\frac{n_+}{2} - \frac{\text{rot}(D)}{2} + \text{rot}(p)} t^{-\frac{\text{rot}(D)}{2} + \frac{\text{rot}(p)}{2}} Q(t)$$

satisfies Alexander-Conway skein relation

$$\nabla(\quad)(t) - \nabla(\quad)(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\nabla(\quad)(t),$$

and is invariant under Reidemeister moves.

Bibliography

[1] D. Allcock. Braid pictures for Artin groups. *Trans. Amer. Math. Soc.*, 354(9):3455–3474, 2002.

[2] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. Homotopy classification of ribbon tubes and welded string links. to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci., 2015.

[3] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. On forbidden moves and the delta move. arXiv:1510.04237, 2015.

[4] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. On usual, virtual and welded knotted objects up to homotopy. to appear in J. Math. Soc. Japan, 2015.

[5] A. Bartels and P. Teichner. All two dimensions links are null homotopic. *Geom. Topol.*, 3:235–252, 1999.

[6] J. Birman and H. Wenzl. Braids, link polynomials and a new algebra. *Trans. Amer. Math. Soc.*, 313:249–273, 1989.

[7] M. Broué and G. Malle. Zyklotomische Heckealgebren. *Astérisque*, pages 119–189, 1993.

[8] R. D. Budney. On the image of the Lawrence-Krammer representation. *J. Knot Theory Ramifications*, 14(6):773–789, 2005.

[9] R. Charney and D. Peifer. The $K(\pi, 1)$-conjecture for the affine braid groups. *Comment. Math. Helv.*, 78(3):584–600, 2003.

[10] H. Coxeter. Factor groups of the braid groups. *Proc. Fourth Canad. Math. Congress*, pages 95–122, 1957.

[11] J.-M. Droz and E. Wagner. Grid diagrams and Khovanov homology. *Algebr. Geom. Topol.*, 9(3), 2009.

[12] M. Eisermann. The Jones polynomial of ribbon links. *Geom. Topol.*, 13:623–660, 2009.

[13] R. H. Fox. Some problems in knot theory. In *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, pages 168–176. Prentice-Hall, Englewood Cliffs, N.J., 1962.

[14] A. Gadbled, A.-L. Thiel, and E. Wagner. Categorical action of the extended affine braid group of type A. to appear in Commun. Contemp. Math., 2015.

[15] N. Habegger and X.-S. Lin. The classification of links up to link-homotopy. *J. Amer. Math. Soc.*, 3:389–419, 1990.

[16] N. Habegger and X.-S. Lin. On link concordance and Milnor's $\overline{\mu}$ invariants. *Bull. London Math. Soc.*, 30(4):419–428, 1998.

[17] N. Habegger and J.-B. Meilhan. On the classification of links up to finite type. In *Topology and physics*, volume 12 of *Nankai Tracts Math.*, pages 138–150. World Sci. Publ., Hackensack, NJ, 2008.

[18] R. Huerfano and M. Khovanov. A category for the adjoint representation. *J. Algebra*, 246(2):514–542, 2001.

[19] A. Ishii. Algebraic links and skein relations of the Links-Gould invariant. *Proc. Amer. Math. Soc.*, 132(12):3741–3749 (electronic), 2004.

[20] A. Ishii. The Links-Gould invariant of closed 3-braids. *J. Knot. Theory Ramifications*, 13:41–56, 2004.

[21] A. Ishii. The Links-Gould polynomial as a generalization of the Alexander-Conway polynomial. *Pacific J. Math.*, 225:273–285, 2006.

[22] A. Ishii, K. Ueda, and H. Uehara. Stability conditions on A_n-singularities. *J. Differential Geom.*, 84(1):87–126, 2010.

[23] T. Kanenobu and A. Shima. Two filtrations of ribbon 2-knots. In *Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999)*, volume 121, pages 143–168, 2002.

[24] R. Kent, IV and D. Peifer. A geometric and algebraic description of annular braid groups. *Internat. J. Algebra Comput.*, 12(1-2):85–97, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[25] M. Khovanov, V. Mazorchuk, and C. Stroppel. A brief review of abelian categorifications. *Th. Appl. Categ.*, 22(19):479–508, 2009.

[26] M. Khovanov and L. Rozansky. Virtual crossings, convolutions and a categorification of the so(2n) Kauffman polynomial. *J. Gökova Geom. Topol. GGT*, 1:116–214, 2007.

[27] M. Khovanov and L. Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008.

[28] M. Khovanov and L. Rozansky. Matrix factorizations and link homology. II. *Geom. Topol.*, 12(3):1387–1425, 2008.

[29] M. Khovanov and P. Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15(1):203–271, 2002.

[30] G. Lehrer and R. Zhang. Strongly multiplicity free modules for Lie algebras and quantum groups. *J. Algebra*, 306:138–174, 2006.

[31] R. Lipshitz, P. S. Ozsváth, and D. P. Thurston. A faithful linear-categorical action of the mapping class group of a surface with boundary. *J. Eur. Math. Soc. (JEMS)*, 15(4):1279–1307, 2013.

[32] M. Mackaay, M. Stošić, and P. Vaz. A diagrammatic categorification of the q-Schur algebra. *Quantum Topol.*, 4:1–75, 2013.

[33] G. Malle and J. Michel. Constructing representations of Hecke algebras for complex reflection groups. *LMS J. Comput. Math.*, pages 426–450, 2010.

[34] I. Marin. *Représentations linéaires des tresses infinitésimales*. PhD thesis, Université Paris-11, 2001.

[35] I. Marin. Artin groups and Yokonuma-Hecke algebras. arXiv:1601.03191, 2016.

[36] I. Marin and E. Wagner. Markov traces on the Birman-Wenzl-Murakami algebras. arXiv:1403.4021.

[37] I. Marin and E. Wagner. A cubic defining algebra for the Links-Gould polynomial. *Advances in Math.*, 248:1332–1365, 2013.

[38] V. Mazorchuk and C. Stroppel. Categorification of (induced) cell modules and the rough structure of generalised Verma modules. *Adv. Math.*, 219(4):1363–1426, 2008.

[39] H. Morton and A. Wassermann. A basis for the Birman-Wenzl algebra. arXiv:1012.3116, 1989.

[40] H. Murakami, T. Ohtsuki, and S. Yamada. Homfly polynomial via an invariant of colored plane graphs. *Enseign. Math. (2)*, 44(3-4):325–360, 1998.

[41] J. Murakami. The Kauffman polynomial of links and representation theory. *Osaka J. Math.*, 24:745–758, 1987.

[42] L. Poulain d'Andecy, A.-L. Thiel, and E. Wagner. Classical and transverse Markov traces on the BMW algebra. in preparation, 2016.

[43] L. Poulain d'Andecy and E. Wagner. The HOMFLYPT polynomials of sublinks and the Yokonuma-Hecke algebras. arXiv:1606.00237, 2016.

[44] C. M. Ringel. *Tame algebras and integral quadratic forms*, volume 1099 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[45] D. Rose. A Note on the Grothendieck Group of an Additive Category. math.CT/1109.2040.

[46] S. Satoh. Virtual knot presentation of ribbon torus-knots. *J. Knot Theory Ramifications*, 9(4):531–542, 2000.

[47] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[48] W. Soergel. Kazhdan-Lusztig-polynome und unzerlegbare bimoduln über polynomringen. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.

[49] C. C. Squier. The Burau representation is unitary. *Proc. Amer. Math. Soc.*, 90(2):199–202, 1984.

[50] J. Stallings. Homology and central series of groups. *J. Algebra*, 2:170–181, 1965.

[51] S. Suzuki. Knotting problems of 2-spheres in 4-sphere. *Math. Sem. Notes Kobe Univ.*, 4(3):241–371, 1976.

[52] T. tom Dieck. Categories of rooted cylinder ribbons and their representations. *J. Reine Angew. Math.*, 494:35–63, 1998.

[53] P. Vaz and E. Wagner. A remark on BMW algebra, q-Schur algebras and categorification. *Canad. J. Math.*, 66(2):453–480, 2014.

[54] T. Yajima. On the fundamental groups of knotted 2-manifolds in the 4-space. *J. Math. Osaka City Univ.*, 13:63–71, 1962.

[55] T. Yajima. On simply knotted spheres in R^4. *Osaka J. Math.*, 1:133–152, 1964.

[56] T. Yanagawa. On ribbon 2-knot: The 3-manifold bounded by the 2-knots. *Osaka J. Math.*, 6:447–464, 1969.