Khovanov-Rozansky homology for embedded graphs

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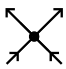
February 2, 2009

Abstract

For any positive integer n, Khovanov and Rozansky constructed a bigraded link homology from which you can recover the $\mathfrak{sl}_n$ link polynomial invariants. We generalize Khovanov-Rozansky construction in the case of finite 4-valent graphs embedded into a ball $B^3 \subset \mathbb{R}^3$. More precisely, we prove that the homology associated to a diagram of a 4-valent graph embedded in $B^3 \subset \mathbb{R}^3$ is invariant under the graph moves introduced by Kauffman.

Introduction

We consider finite oriented 4-valent graphs embedded into a ball $B^3 \subset \mathbb{R}^3$. We fix a great circle on the boundary 2-sphere of B^3 and require that the boundary points of the embedded graph lie on this great circle and that the orientations around a vertex is as follows

.

These graphs are called *open regular graphs*. A *diagram* Γ of an open regular graph is a generic projection of the graph onto the plane of the great circle. An isotopy of such a graph should not move its boundary points and respect the cyclic order on the vertices. An embedded graph into B^3 without boundary points is called a *(closed) regular graph*. If an open regular graph can be embedded into the plane of the great circle, it is called *planar* and we make no distinction between the graph and this generic projection, see Figure 1 for an example.

For any positive integer n, Khovanov and Rozansky categorified the $\mathfrak{sl}_n$ link polynomials [4] by associating to a link diagram a complex of matrix factorizations (see section 1.1 for a definition of matrix factorization). The first step in their construction, is to associate a matrix factorization to an open planar regular graph. Using their construction it is almost immediate that one can associate a complex of matrix factorizations to a diagram of an

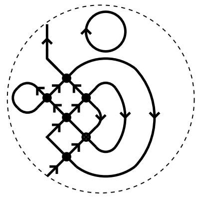


Figure 1: Planar regular graph

open regular graph. We denote by $C_n(\Gamma)$ the complex of matrix factorization associated to a diagram Γ of an open regular graph. In order to prove that the homotopy type of this complex of matrix factorizations is an invariant of the open regular graph, we check that it is invariant under the graph moves, called RV-moves introduced by Kauffman [2], see Figure 4. Hence, we obtain the following theorem:

Theorem 1 *Let $C_n(\Gamma_1)$ and $C_n(\Gamma_2)$ be complexes of matrix factorizations associated to Γ_1 and Γ_2 diagrams of open regular graphs in B^3. If there exist a sequence of RV-moves such that Γ_2 is obtained from Γ_1 then $C_n(\Gamma_1)$ and $C_n(\Gamma_2)$ are homotopy equivalent.*

As pointed out by Kauffman and Vogel [3], link polynomial invariants give rise to graph invariants, the same is true for Khovanov-Rozansky link homology.

In section 1, we recall the Khovanov-Rozansky construction and adapt it to the case of oriented 4-valent graphs embedded in $B^3 \subset \mathbb{R}^3$. In section 2, we introduce Kauffman graph moves and in section 3, we prove the invariance up to homotopy of the complex of matrix factorizations under these moves.

1 Khovanov-Rozansky construction

1.1 Matrix factorizations

Let k be a positive integer, $R = \mathbb{Q}[x_1, \cdots, x_k]$ be a commutative polynomial $\mathbb{Q}$-algebra and $w \in R$. A (R, w)-*matrix factorization* of potential w over R consists of two free R-modules C^0, C^1 and two R-module maps

$$C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^0$$

such that $d^0 \circ d^1(m) = wm$ for all $m \in C^1$ and $d^1 \circ d^0(m) = wm$ for all $m \in C^0$.

A first example of matrix factorization is the following (R, ab)-matrix factorization

$$R \xrightarrow{\times a} R \xrightarrow{\times b} R,$$

where $a, b \in R$. We note this matrix factorization $(a, b)_R$. We construct more matrix factorizations by tensoring over R such elementary factorizations (see [4] for a definition of tensor products). More precisely we denote by

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix}_R$$

the tensor product over R of $(a_1, b_1)_R$, $(a_2, b_2)_R$,..., $(a_k, b_k)_R$. It's a matrix factorization of potential $w = a_1 b_1 + \cdots + a_k b_k$. We consider the following $\mathbb{Z}$-grading on R: $deg(x_i) = 2$, for $i = 1, \ldots, k$. A matrix factorization is graded if d^0 and d^1 are homogenous and $deg(d^0) = deg(d^1)$. The grading on R induces a grading on C^0 and C^1: $C^0 = \oplus_{i \in \mathbb{Z}} C^{i,0}$, $C^1 = \oplus_{i \in \mathbb{Z}} C^{i,1}$, We denote by curly bracket$\{\ .\ \}$ the shift of the $\mathbb{Z}$-grading: for $i, k \in \mathbb{Z}$ and $j \in \mathbb{Z}/2\mathbb{Z}$, $C^{i,j}\{k\} = C^{i-k,j}$. For $k \in \mathbb{Z}$, denote $\langle k \rangle$ the shift of the $(\mathbb{Z}/2\mathbb{Z})$-grading by k (mod 2). Given two graded (R, w)-matrix factorizations C and D, a *morphism* $f : C \to D$ is a pair of R-module homomorphisms $f^0 : C^0 \to D^0$ and $f^1 : C^1 \to D^1$ such that the following diagram commutes,

$$\begin{array}{ccccc} C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^0 \\ \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^0} \\ D^0 & \xrightarrow{\delta^0} & D^1 & \xrightarrow{\delta^1} & D^0 \end{array}$$

The R-module homomorphisms f^0, f^1 preserve the $\mathbb{Z}$-grading. A *homotopy* h between morphisms $f, g : C \to D$ of matrix factorizations is a pair of morphisms $h^0 : C^0 \to D^1$ and $h^1 : C^1 \to D^0$ such that

$$f^0 - g^0 = h^1 \circ d^0 + \delta^1 \circ h^0, \text{ and } f^1 - g^1 = h^0 \circ d^1 + \delta^0 \circ h^1.$$

Given $w \in R$, we denote by hmf_w^R the homotopy category of graded matrix factorizations of potential w over R.

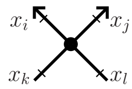
1.2 Planar regular graphs and matrix factorizations

Fix a positive integer n. We now recall the matrix factorizations that Khovanov and Rozansky [4] associate to an open planar regular graph. Let Γ be an open planar regular graph, k a positive integer and $\{x_1, \ldots, x_k\}$ a set of marks on the edges of Γ such that every edge has a least one mark.
We consider the following piece of a planar regular graph

The matrix factorization L_j^i associated to this piece is $(\pi_{ij}, x_i - x_j)_{\mathbb{Q}[x_i,x_j]}$:

$$\mathbb{Q}[x_i, x_j] \xrightarrow{\times \pi_{ij}} \mathbb{Q}[x_i, x_j]\{1-n\} \xrightarrow{\times(x_i - x_j)} \mathbb{Q}[x_i, x_j],$$

where $\pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}$. The grading shift makes L_j^i graded. More precisely with this shift multiplications by π_{ij} and $x_i - x_j$ become homogeneous maps of degree $n+1$. We now consider the following other piece Γ^1 of a planar regular graph:



Fix R to be the ring $\mathbb{Q}[x_i, x_j, x_k, x_l]$. We associate to such a piece a graded matrix factorization of potential $w = {x_i}^{n+1} + {x_j}^{n+1} - {x_k}^{n+1} - {x_l}^{n+1}$ over R. We decompose w,

$$w = u_1(x_i, x_j, x_k, x_l)(x_i + x_j - x_k - x_l) + u_2(x_i, x_{j,k}, x_l)(x_i x_j - x_k x_l)$$

where u_1 and u_2 are determined by the following relation:

$$u_1 = \frac{g(x_i + x_j, x_i x_j) - g(x_k + x_l, x_i x_j)}{x_i + x_j - x_k - x_l},$$

$$u_2 = \frac{g(x_k + x_l, x_i x_j) - g(x_k + x_l, x_k x_l)}{x_i x_j - x_k x_l},$$

and g is the two variable function satisfing $g(x+y, xy) = x^{n+1} + y^{n+1}$. We now define $C_n(\Gamma^1)$ to be the graded matrix factorization

$$\begin{pmatrix} u_1 & x_i + x_j - x_k - x_l \\ u_2 & x_i x_j - x_k x_l \end{pmatrix}_R \{-1\}$$

In other words $C_n(\Gamma^1)$ is the tensor product over R of the graded matrix factorizations

$$R \xrightarrow{u_1} R\{1-n\} \xrightarrow{x_i + x_j - x_k - x_l} R$$

and

$$R \xrightarrow{u_2} R\{3-n\} \xrightarrow{x_i x_j - x_k x_l} R$$

with an additionnal shift by $\{-1\}$. As for L_i^j, the grading shift makes $C_n(\Gamma^1)$ graded.

We distinguish two kinds of tensor products of such elementary graded matrix factorizations: tensor product over $\mathbb{Q}$ corresponding topologically to a disjoint union of pieces and tensor products over some polynomial $\mathbb{Q}$-algebra corresponding to a gluing of pieces along some endpoints, see [4]

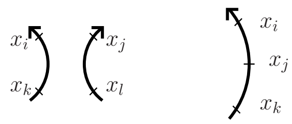


Figure 2: Two examples

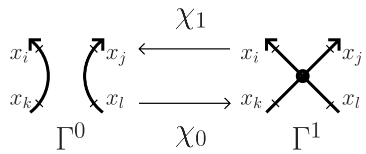
for a detailed treatment. The potential of graded matrix factorizations is additive with respect to both tensor products. We consider the two examples on Figure 2. The graded matrix factorization associated to the left diagram of Figure 2 is $L_k^i \otimes_{\mathbb{Q}} L_l^j$ and the graded matrix factorization associated to the right diagram in Figure 2 is $L_j^i \otimes_{\mathbb{Q}[x_j]} L_k^j$. In general, we can now associate to a planar regular graph Γ embedded in $\mathbb{R}^2$,

$$C_n(\Gamma) = \left(\otimes L_i^j\right) \bigotimes \left(\otimes C_n(\Gamma^1)\right)$$

where the first tensor runs through all the oriented arcs starting and ending at marks with no interior mark, and the second runs trough all 4-valent vertices. The tensor products are over suitable polynomial $\mathbb{Q}$-algebras, see [4]. The homotopy type of this matrix factorization does not depend on the choice of marks [4].

1.3 Regular graph embedded in $\mathbb{R}^3$ and complex of graded matrix factorizations

We define two morphisms χ_0 and χ_1 of graded matrix factorizations between elementary matrix factorizations as depicted on the following diagram:



The matrix factorization $C_n(\Gamma^0)$ is the tensor product over $\mathbb{Q}$ of L_k^i and L_l^j and is given by

$$\begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix} \xrightarrow{P_0} \begin{pmatrix} R\{1-n\} \\ R\{1-n\} \end{pmatrix} \xrightarrow{P_1} \begin{pmatrix} R \\ R\{2-2n\} \end{pmatrix}$$

where

$$P_0 = \begin{pmatrix} \pi_{ik} & x_j - x_l \\ \pi_{jl} & x_k - x_i \end{pmatrix}, \; P_1 = \begin{pmatrix} x_i - x_k & x_j - x_l \\ \pi_{jl} & -\pi_{ik} \end{pmatrix}.$$

The matrix factorization $C_n(\Gamma^1)$ is

$$\begin{pmatrix} R\{-1\} \\ R\{3-2n\} \end{pmatrix} \xrightarrow{Q_0} \begin{pmatrix} R\{-n\} \\ R\{2-n\} \end{pmatrix} \xrightarrow{Q_1} \begin{pmatrix} R\{-1\} \\ R\{3-2n\} \end{pmatrix}$$

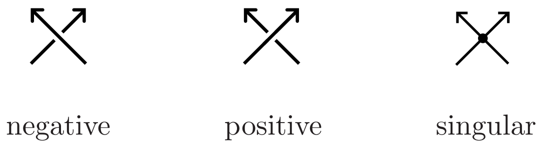


Figure 3: Crossings

where

$$Q_0 = \begin{pmatrix} u_1 & x_ix_j - x_kx_l \\ u_2 & -x_i - x - j + x_k + x_l \end{pmatrix}$$

$$Q_1 = \begin{pmatrix} x_i + x_j - x_k - x_l & x_ix_j - x_kx_l \\ u_2 & -u_1 \end{pmatrix}.$$

We define $\chi_0 : C_n(\Gamma^0) \to C_n(\Gamma^1)$ by the pair of matrices

$$U_0 = \begin{pmatrix} x_k - x_j & 0 \\ a & 1 \end{pmatrix}, U_1 = \begin{pmatrix} x_k & -x_j \\ -1 & 1 \end{pmatrix}$$

acting on $C_n^0(\Gamma^0)$ and $C_n^1(\Gamma^0)$ respectively. The morphism $\chi_1 : C_n(\Gamma^1) \to C_n(\Gamma^0)$ is defined by the pair of matrices

$$V_0 = \begin{pmatrix} 1 & 0 \\ -a & x_k - x_j \end{pmatrix}, V_1 = \begin{pmatrix} 1 & x_j \\ 1 & x_k \end{pmatrix}$$

acting on $C_n^0(\Gamma^1)$ and $C_n^1(\Gamma^1)$ respectively, where $a = -u_2 + (u_1 + x_iu_2 - \pi_{jl})/(x_i - x_k)$. The maps χ_0 and χ_1 are morphisms of graded matrix factorizations and are of degree 1 (for the grading $\{.\}$).
We now consider a regular graph embedded in $\mathbb{R}^3$. We note D a diagram for this graph. It has three different types of crossing: positive, negative and singular, see Figure 1.3.

Let k, r, and s be positive intergers. We put marks $\{x_1, \ldots, x_k\}$ on D such that every arc between two crossings has at least one mark. We put also marks $\{p_1, \ldots, p_r\}$ on every positive or negative crossing and $\{q_1, \ldots, q_s\}$ on every singular crossing. As for planar regular graphs we want associate to every elementary piece of a regular graph diagram an algebraic object, and in this case it is a complex of graded matrix factorizations.

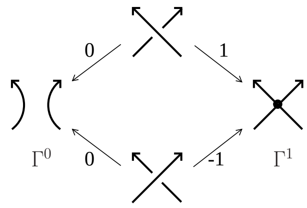
For an arc that contains no crossings and no other marks, define L_j^i as above and consider it as the chain complex

$$0 \longrightarrow L_j^i \longrightarrow 0.$$

where L_i^j is in cohomological degree 0.
For a singular crossing q, define $C_n(q)$ as $C_n(\Gamma^1)$ and consider it as the chain complex

$$0 \longrightarrow C_n(\Gamma^1) \longrightarrow 0.$$

where $C_n(\Gamma^1)$ is in cohomological degree 0. We consider now positive and negative crossings. For every positive or negative crossing p, there is two different resolutions, either Γ^0 or Γ^1.



If p^- is a negative crossing we define $C_n(p^-)$ to be the chain complex

$$0 \longrightarrow C_n(\Gamma^0)\{1-n\} \xrightarrow{\chi_0} C_n(\Gamma^1)\{-n\} \longrightarrow 0,$$

and if p^+ is a positive crossing we define $C_n(p^+)$ to be the chain complex

$$0 \longrightarrow C_n(\Gamma^1)\{n\} \xrightarrow{\chi_1} C_n(\Gamma^0)\{n-1\} \longrightarrow 0,$$

where $C_n(\Gamma^0)$ is always in cohomological degree 0.
Now to a regular graph diagram D associate the complex of graded matrix factorizations

$$C_n(D) = \left(\bigotimes_{L^i_j} L^i_j\right) \bigotimes \left(\bigotimes_p C_n(p)\right) \bigotimes \left(\bigotimes_q C_n(q)\right)$$

where the first tensor runs through all arcs in D starting and ending at marks that contain no crossings and no other marks, p runs through all the positive and negative crossings of D and q runs through all singular crossings. The tensor products are over suitable polynomial $\mathbb{Q}$-algebras.

2 Reidemeister moves for graphs

We consider open regular graphs embedded in $B^3 \subset \mathbb{R}^3$ as graphs with rigid vertices. As explained in [2], a 4-valent graph with rigid vertices can be regarded as an embedding of a graph whose vertices have been replaced by rigid disks. Each disk has four strands attached to it, and the cyclic order of these strands is determined via the rigidity of the disk. An *RV-isotopy or rigid vertex isotopy* of the embedding of such a regular graph Γ in $\mathbb{R}^3$ consists in affine motions of the disks, coupled with topological ambient isotopies of the strands (corresponding to the edge of Γ). The notion of RV-isotopy is a mixture of mechanical (Euclidian) and topological concepts. It arise naturally in the building of models for graph embeddings, and it also arises naturally in regard to creating invariant of graph embeddings.
In [2], Kauffman derived a collection of moves, anagolous to Reidemeister

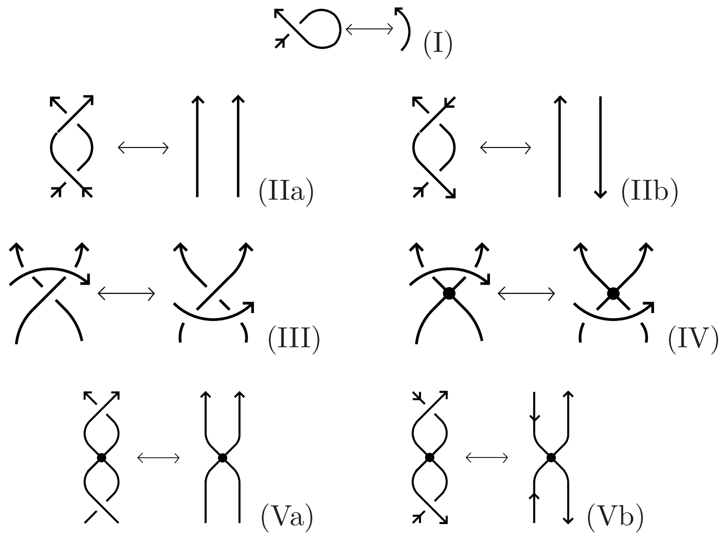
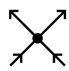


Figure 4: Graph moves that generate rigid vertex isotopy

moves, that generates RV-isotopy for diagrams of 4-valent graph embeddings. As we are only interrested in 4-valent oriented graph embeddings whose oriented rigid vertex take the basic form



we will present the RV-moves in this case, see Figure 4.

3 Invariance under RV-moves

In [4], Khovanov and Rozansky have proved the invariance of $C_n(\Gamma)$ under type (I), (II) and (III) moves, see Figure 4. We prove the invariance under type (IV) and (V). The invariance under type (IV) follows directly from the proof of invariance under (III). We will use at many level the proofs of Khovanov and Rozansky, see [4]. All isomorphisms under graded matrix factorizations below are in homotopy categories hmf.

3.1 Invariance under (IV)

As pointed out by Wu [6], the Khovanov-Rozansky's proof of the invariance under Reidemeister (III) can be simplified by using Bar-Natan's algebraic trick [1], i.e by using the fact that the homotopy equivalence used for the proof of the invariance under Reidemeister move (IIa) is a strong deformation retract. If we think in the proof that way, then the proof of invariance

Figure 5: Type (IV) move

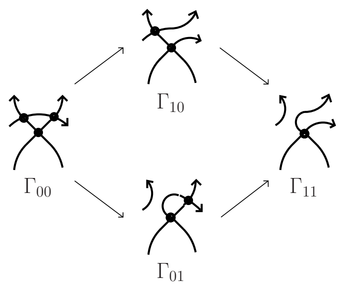


Figure 6: Four resolutions of Γ in the type (IV) move

under (IV) is contained in the one of (III).

We need to show that $C_n(\Gamma)$ and $C_n(\Gamma')$ are isomorphic for Γ, Γ' in Figure 5. The diagram Γ has 4 resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$, see Figure 6. The complex $C_n(\Gamma)\{-2n\}$ has the form

$$0 \to C_n(\Gamma_{00}) \stackrel{\partial^{-2}}{\to} \begin{pmatrix} C_n(\Gamma_{01})\{-1\} \\ C_n(\Gamma_{10})\{-1\} \end{pmatrix} \stackrel{\partial^{-1}}{\to} C_n(\Gamma_{11})\{-2\} \to 0$$

with $C_n(\Gamma_{11})\{-2\}$ in cohomological degree 0. This complex is shown in Figure 6.

Khovanov and Rozansky proved [4] the following isomorphism:

$$C_n(\Gamma_{01}) \cong C_n(\Gamma_{11})\{+1\} \oplus C_n(\Gamma_{11})\{-1\}. \tag{1}$$

Furthermore, they proved that

$$C_n(\Gamma_{00}) \cong C_n(\Gamma_{11}) \oplus \Upsilon, \tag{2}$$

where Υ is defined in [4, Prop. 33].

The differential ∂^{-2} is injective on $C_n(\Gamma_{11}) \subset C_n(\Gamma_{00})$. In fact, the map to $C_n(\Gamma_{01})\{-1\}$ is injective (which follows from the inclusion $C_n(\Gamma_{11}) \subset C_n(\Gamma_{00})$ and the proof of invariance under (IIa), see [4]). The graded matrix factorization $\partial^{-2}(C_n(\Gamma_{00}))$ is a direct summand of $C_n^{-1}(\Gamma)\{-2n\}$. Thus $C_n(\Gamma)\{-2n\}$ contains a contractible summand

$$0 \to C_n(\Gamma_{11}) \stackrel{\partial^{-2}}{\to} C_n(\Gamma_{11}) \to 0. \tag{3}$$

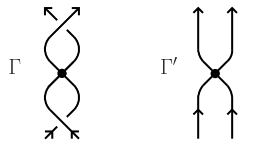


Figure 7: Type (Va) move

The direct sum decomposition (1) can be chosen so that

$$C_n(\Gamma_{01})\{-1\} \cong p_{01}\partial^{-2}C_n(\Gamma_{11}) \oplus C_n(\Gamma_{11})\{-2\},$$

where p_{01} is the projection of $C_n^{-1}(\Gamma)\{-2n\}$ onto $C_n(\Gamma_{01})\{-1\}$. The differential ∂^{-1} is injective on $C_n(\Gamma_{11})\{-2\} \subset C_n(\Gamma_{01})\{-1\}$. Furthermore, the image of $C_n(\Gamma_{11})\{-2\} \subset C_n(\Gamma_{01})\{-1\}$ under ∂^{-1} is a direct summand of $C_n^0(\Gamma)\{-2n\}$. Hence the complex $C_n(\Gamma)\{-2n\}$ contains a contractible direct summand isomorphic to

$$0 \to C_n(\Gamma_{11})\{-2\} \stackrel{\partial^{-1}}{\to} C_n(\Gamma_{11})\{-2\} \to 0 \tag{4}$$

After splitting off contractible direct summands (3) and (4), the complex $C_n(\Gamma)\{-2n\}$ reduces to the complex C defined by

$$0 \to \Upsilon \stackrel{\partial^{-2}}{\to} C_n(\Gamma_{10})\{-1\} \to 0$$

Since both $C_n(\Gamma)\{-2n\}$ and $C_n(\Gamma')\{-2n\}$ contain $C_n(\Gamma_{10})\{-1\}$, [4, Prop. 33] ensures that we can perform exactly the same reduction to $C_n(\Gamma')\{-2n\}$. Finally, we conclude that

$$C_n(\Gamma) \cong C_n(\Gamma').$$

3.2 Invariance under (Va)

This invariance can be obtained as a consequence of Lemma 4.10 from Rasmussen [5]. We detail the proof.

We need to show that $C_n(\Gamma)$ and $C_n(\Gamma')$ are isomorphic for the graphs Γ, Γ' shown in Figure 7. The diagram Γ has 4 resolutions, denoted by Γ_{ij} for $i, j \in \{0, 1\}$ and shown in Figure 8. The complex $C_n(\Gamma)$ has the form

$$0 \to C_n(\Gamma_{00})\{+1\} \stackrel{\partial^{-1}}{\to} \begin{pmatrix} C_n(\Gamma_{01}) \\ C_n(\Gamma_{10}) \end{pmatrix} \stackrel{\partial^{0}}{\to} C_n(\Gamma_{11})\{-1\} \to 0$$

where $C_n(\Gamma_{01})$ and $C_n(\Gamma_{10})$ are in cohomological degree 0. We have depicted this complex in Figure 8. Since Γ_{00} and Γ_{11} are isotopic, $C_n(\Gamma_{00})$ and $C_n(\Gamma_{11})$ are isomorphic. Khovanov and Rozansky [4] proved that

$$C_n(\Gamma_{10}) \cong C_n(\Gamma_{00})\{+1\} \oplus C_n(\Gamma_{00})\{-1\}. \tag{5}$$

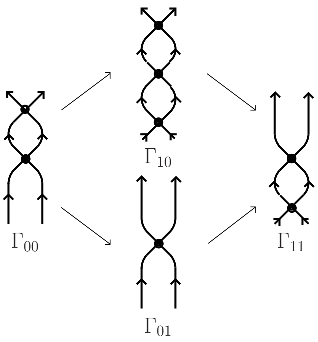


Figure 8: Four resolutions of Γ in the type (Va) move

Khovanov-Rozansky's proof of invariance under (IIa) ensures that the differential ∂^{-1} is injective on $C_n^{-1}(\Gamma_{00})$. Direct sum decomposition (5) can be chosen so that

$$C_n(\Gamma_{10}) \cong p_{10}\partial^{-1}C_n(\Gamma_{00})\{+1\} \oplus C_n(\Gamma_{00})\{-1\}.$$

Thus, $C_n(\Gamma)$ contains a contractible summand

$$0 \to C_n(\Gamma_{00})\{+1\} \stackrel{\partial^{-1}}{\to} C_n(\Gamma_{00})\{+1\} \to 0. \tag{6}$$

Furthermore, we have

$$C_n(\Gamma_{11})\{-1\} \cong C_n(\Gamma_{01}) \oplus C_n(\Gamma_{01})\{-2\}. \tag{7}$$

Differential ∂^0 is surjective onto $C_n(\Gamma_{01}) \subset C_n(\Gamma_{11})\{-1\}$. Thus $C_n(\Gamma)$ contains a contractible summand

$$0 \to C_n(\Gamma_{01}) \stackrel{\partial^0}{\to} C_n(\Gamma_{01}) \to 0. \tag{8}$$

After splitting off the contractible direct summands (6) and (8), the complex $C_n(\Gamma)$ reduces to the complex C of the form

$$0 \to C_n(\Gamma_{11})\{-1\} \stackrel{\partial^0}{\to} C_n(\Gamma_{01})\{-2\} \to 0$$

Since Γ' and Γ_{01} are isotopic, the decomposition (7) ensures that C is homotopy equivalent to

$$0 \to C_n(\Gamma') \to 0.$$

3.3 Invariance under (Vb)

We need to show that $C_n(\Gamma)$ and $C_n(\Gamma')$ are isomorphic for the graphs Γ, Γ' shown in Figure 9. The diagram Γ has 4 resolutions, denoted by Γ_{ij} for

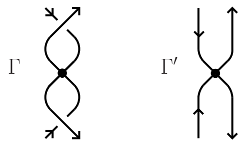


Figure 9: Type (Vb) move

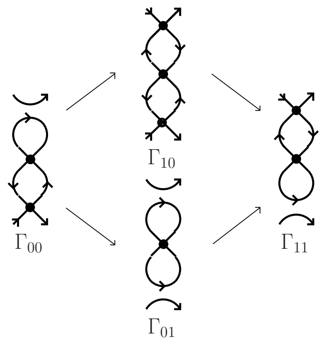


Figure 10: Four resolution of Γ in the type (Vb) move

$i, j \in \{0, 1\}$ and shown in Figure 10. The complex $C_n(\Gamma)$ has the form

$$0 \to C_n(\Gamma_{00})\{+1\} \overset{t(\partial^{-1,0},\partial^{-1,1})}{\to} \begin{pmatrix} C_n(\Gamma_{10}) \\ C_n(\Gamma_{01}) \end{pmatrix} \overset{(\partial^{0,0},\partial^{0,1})}{\to} C_n(\Gamma_{11})\{-1\} \to 0$$

where $C_n(\Gamma_{01})$ and $C_n(\Gamma_{10})$ are in cohomological degree 0. We have depicted this complex in Figure 10.

Applying Khovanov-Rozansky's results, we have the following isomorphisms:

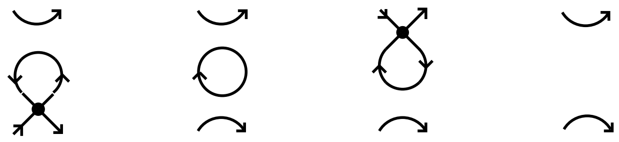
$$C_n(\Gamma_{00}) \cong \bigoplus_{i=0}^{n-2} C_n(G_{00})\{2-n+2i\}\langle 1\rangle, \tag{9}$$

$$C_n(\Gamma_{01}) \cong \bigoplus_{i=0}^{n-2} C_n(G_{01})\{2-n+2i\}\langle 1\rangle, \tag{10}$$

$$C_n(\Gamma_{11}) \cong \bigoplus_{i=0}^{n-2} C_n(G_{11})\{2-n+2i\}\langle 1\rangle, \tag{11}$$

and

$$C_n(\Gamma_{10}) \cong \left(\bigoplus_{i=0}^{n-3} C_n(G_{11})\{3-n+2i\}\langle 1\rangle\right) \oplus C_n(\Gamma'), \tag{12}$$



G_{00} G_{01} G_{11} G

Figure 11: The graph G_{00}, G_{01}, G_{11}, G

where G_{00}, G_{01} and G_{11} are depicted in Figure 11.

We can twist the direct sum decompositions (9), (10) and (11) so that $\partial^{-1,1}$ and $\partial^{0,1}$ have diagonal form following this decomposition: $\partial^{-1,1} = \sum_{i=0}^{n-2} \partial_i^{-1,1}$ and $\partial^{0,1} = \sum_{i=0}^{n-2} \partial_i^{0,1}$. The proof of invariance under (I) by Khovanov and Rozansky implies that $\partial_i^{-1,1}$ is split injective and $\partial_i^{0,1}$ is split surjective for all $i \in [\![0, n-2]\!]$. Hence we have that $\partial^{-1,1}$ is split injective and $\partial^{0,1}$ is split surjective. Denote δ_i^{-1} the restriction of δ^{-1} to $C_n(G_{00})\{3-n+2i\}\langle 1\rangle$ and δ_i^0 the composition of δ^0 with the projection onto $C_n(G_{11})\{1-n+2i\}\langle 1\rangle$. Since the category hmf_w has splitting idempotents (see [4, p. 46]), we can decompose $C_n^0(\Gamma)$ as the direct sum

$$C_n^0(\Gamma) \cong \left(\bigoplus_{i=0}^{n-2} Im(\partial_i^{-1})\right) \oplus \left(\bigoplus_{i=0}^{n-2} Y_1^i\right) \oplus Y_2$$

in such a way that ∂_i^0 restrict to an isomorphism from Y_1^i to $C_n(G_{11})\{1-n+2i\}$ for all $i = 0, \ldots, n-2$ and $\partial_i^0(Y_2) = 0$. Therefore, $C_n(\Gamma)$ is isomorphic to the direct sum of complexes

$$\begin{array}{ccccccccc} & & 0 & \rightarrow & Y_2 & \rightarrow & 0, & & \\ 0 & \rightarrow & C_n(G_{00})\{3-n+2i\} & \xrightarrow{\cong} & \mathrm{Im}(\partial_i^{-1}) & \rightarrow & 0, & & \\ & & 0 & \rightarrow & Y_1^i & \xrightarrow{\cong} & C_n(G_{11})\{1-n+2i\} & \rightarrow & 0. \end{array}$$

We can decompose further the sum decompositions (9), (10) and (11) so that we obtain

$$C_n(\Gamma_{00}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-2} C_n(G)\{4-2n+2(i+j)\}, \tag{13}$$

$$C_n(\Gamma_{01}) \cong \bigoplus_{i=0}^{n-2} \bigoplus_{j=0}^{n-1} C_n(G)\{3-2n+2(i+j)\}, \tag{14}$$

$$C_n(\Gamma_{11}) \cong \bigoplus_{i=0}^{n-2}\bigoplus_{j=0}^{n-2} C_n(G)\{4-2n+2(i+j)\}, \tag{15}$$

and

$$C_n(\Gamma_{10}) \cong \left(\bigoplus_{i=0}^{n-3}\bigoplus_{j=0}^{n-2} C_n(G)\{5-2n+2(i+j)\}\right) \oplus C_n(\Gamma'), \tag{16}$$

where G is the right-most graph depicted in Figure 11. From formula (13) to (16) we obtain

$$C_n^0(\Gamma) \cong C_n(\Gamma_{01}) \oplus C_n(\Gamma_{10}) \cong C_n(\Gamma_{00})\{+1\} \oplus C_n(\Gamma_{11})\{-1\} \oplus C_n(\Gamma').$$

Category hmf_w is Krull-Schmidt; it implies that $Y_2 \cong C_n(\Gamma')$. Therefore, the complexes $C_n(\Gamma)$ and $0 \to C_n(\Gamma') \to 0$ are isomorphic. This concludes our proof of the invariance under type (Vb) move. Theorem 1 follows.

Aknowledgements This paper is done as part of the author's PhD research and the author wishes to thank Prof. Turaev for his endless support. The author thanks the University of Liverpool for their hospitality.

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